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Turbulence in active fluids caused by self-propulsion

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Abstract. A rigorous analytical justification of turbulence observed in active fluids and caused by self-propulsion is presented. We prove existence of unstable wave modes for the generalized Stokes and Navier–Stokes systems by developing an approach in spaces of Fourier transformed Radon measures.

Keywords: Living fluids, active turbulence, generalized Navier–Stokes equations, well-posedness, stability

1. Introduction

In this brief note we study analytical properties of the following minimal hydrodynamic model to describe the bacterial velocity in the case of highly concentrated bacterial suspensions with negligible density fluctuations considered on the domain $(0, \infty) \times \mathbb{R}^n$:

$$\begin{aligned} &v_t + \lambda_0 v \cdot \nabla v = f - \nabla p + \lambda_1 \nabla |v|^2 - \big(\alpha + \beta |v|^2\big) v + \Gamma_0 \Delta v - \Gamma_2 \Delta^2 v, \\ &\operatorname{div} v = 0, \\ &v(0) = v_0. \end{aligned} \tag{1}$$

Here v is the bacterial velocity field and p the (scalar) pressure and $\lambda_0, \lambda_1, \alpha, \beta, \Gamma_0$ and Γ_2 are real parameters. For $\lambda_0 = 1$, $\lambda_1 = \alpha = \beta = \Gamma_2 = 0$ and $\Gamma_0 > 0$, the model reduces to the incompressible Navier–Stokes equations in n spatial dimensions. For non-vanishing $\lambda_1, \alpha, \beta, \Gamma_2$ system (1) serves as a model to describe occuring turbulence in low Reynolds regimes caused by self-propulsion. It was originally proposed by Wensink et al. in [24] and then considered in Refs. [4,5] and is by now one of the standard models to describe active turbulence at low Reynolds number [20]. The model was recently derived from more microscopic descriptions [13] and was quantitatively confirmed in suspensions of living biological systems [1,15,24,25] and synthetic microswimmers [11]. Last not least, active turbulence was also suggested as a power source for various microfluidic applications [15–17,22]. We refer to those papers and to [26] for a more detailed description of the physics behind the additional occuring terms.

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In [26] an analytical approach to (1) in $L^2(\mathbb{R}^n)$ is presented. The aim of this note is to prove well-posedness and significant results on stability and instability (turbulence) in the $\mathrm{FM}(\mathbb{R}^n)$-setting, i.e., in spaces of Fourier transformed Radon measures. The purpose is to mathematically confirm the asymptotic behavior observed in simulations and experiments as well as the following 'formal' linear stability analysis given in [24]: For $p_0 \in \mathbb{R}$ consider the steady state $(0, p_0)$ of (1) corresponding to a *disordered isotropic state* (see (3)). Plugging the wave ansatz

$$(v, p) := (0, p_0) + (\varepsilon, \eta)\exp(ik \cdot x + \sigma t), \quad k \neq 0, x \in \mathbb{R}^n, t \geqslant 0, \sigma \in \mathbb{R}, \tag{2}$$

with small ε, η into system (1) and neglecting the nonlinear terms yields the characteristic spectral values

$$\sigma(k) = -\big(\alpha + \Gamma_0 k^2 + \Gamma_2 k^4\big).$$

Thus unstable (turbulent) modes are expected to exist for $\Gamma_0 < 0$ and $4\alpha < \Gamma_0^2/\Gamma_2$, or for $\Gamma_0 \geqslant 0$ and $\alpha < 0$. A similar formal argument leads to stable and unstable modes for a manifold of ordered polar states (see also the discussion before Proposition 3.6).

In [26] precise and rigorous results on linear and nonlinear stability and instability in the $L^2(\mathbb{R}^n)$-setting are given, depending on the values of the involved parameters. This, however, does not rigorously confirm the formal stability analysis above just by the fact that the wave ansatz (2) is not an $L^2(\mathbb{R}^n)$-function. (As it is well known, changing the function space, i.e., the functional setting, in general changes the spectrum, the growth bound and their relation.) On the other hand, it is easy to see that (v, p) as given in (2) can be regarded as a Fourier transformed Radon measure, that is, it belongs to the space $\mathrm{FM}(\mathbb{R}^n)$ (see Remark 2.5). In this note we derive precise and rigorous results on linear and nonlinear stability and instability in the $\mathrm{FM}(\mathbb{R}^n)$-setting which justifies the formal argument based on the wave modes (2).

Note that in the context of evolution equations the formal stability analysis given above based on wave modes of the form (2) is standard in applied literature. The approach in FM-spaces to confirm this argument in unbounded domains such as $\mathbb{R}^n$, half-spaces or layers is developed in [9]. It is, e.g., also succesfully applied to confirm stability of the Ekman spiral for low Reynolds numbers in [10] and instability of the Ekman spiral for high Reynolds numbers in [6].

We organized this note as follows. In Section 2 we briefly recall basic facts on the space FM. The main part Section 3 is divided in several subsections. In Section 3.1 we give precise information on linear (in-) stability of the steady states depending on the values of the involved parameters. In Section 3.2 we prove well-posedness for the generalized Navier–Stokes equations (1) in the FM-setting. In fact, we prove existence of a unique maximal strong solution for arbitrary data and existence of a unique global mild solution for small data. In Section 3.3 we transfer most of the results on linear (in-) stability to the nonlinear system (1).

2. The space of Fourier transformed Radon measures

We start with basic notation. For a domain $\Omega \subset \mathbb{R}^n$ and a Banach space X in the sequel $L^p(\Omega, X)$, $1 \leqslant p \leqslant \infty$, denotes the standard Bochner–Lebesgue space with norm

$$\|u\|_{L^p(X)} = \left(\int_\Omega \|u(x)\|_X^p \, dx\right)^{1/p},$$

if $1 \leqslant p < \infty$ and $\|u\|_{L^\infty(X)} = \operatorname{ess\,sup}_{x\in\Omega} \|u(x)\|_X$ in case that $p = \infty$. The space of bounded and continuous functions is denoted $\mathrm{BC}(\Omega, X)$ and we write $\mathrm{BUC}(\Omega, X)$, if the functions are additionally uniformly continuous. As usual, $C_c^\infty(\Omega, X)$ stands for the space of smooth compactly supported functions.

The symbol $W^{k,p}(\Omega, X)$, $k \in \mathbb{N}_0$, $1 \leqslant p \leqslant \infty$, denotes the standard Sobolev space of k-times weak differentiable functions in $L^p(\Omega, X)$. Its norm is given as

$$\|f\|_{W^{k,p}(X)} := \left(\sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|_{L^p(X)}^p\right)^{1/p}$$

with the usual modification if $p = \infty$. The class of all bounded and linear operators from the space X into the space Y we denote by $\mathscr{L}(X, Y)$, where we write $\mathscr{L}(X)$ if $X = Y$, and $\sigma(A)$ denotes the spectrum of a linear operator $A : D(A) \subset X \to X$.

We outline properties of the space of Fourier transformed Radon measures $\mathrm{FM}(\mathbb{R}^n)$. For a comprehensive and detailed introduction we refer to [9].

Definition 2.1. Let $\mathscr{A}$ be a σ-Algebra over $\mathbb{R}^n$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let $\mathbb{K}^m$ be equipped with the euclidian norm $|\cdot|$. A set map $\mu : \mathscr{A} \to \mathbb{K}^m$ is called a finite vector valued Radon measure if

(1) μ is a $\mathbb{K}^m$-valued measure, that is, if $\mu(\emptyset) = 0$ and μ is σ-additive;
(2) the variation of μ defined as

$$|\mu|(\mathcal{O}) := \sup\left\{\sum_{E\in\Pi(\mathcal{O})} \big|\mu(E)\big| : \Pi(\mathcal{O}) \subseteq \mathscr{A} \text{ finite decomposition of } \mathcal{O}\right\}$$

for $\mathcal{O} \in \mathscr{A}$ is a finite Radon measure (that is if $|\mu|(\mathbb{R}^n) < \infty$ and $|\mu|$ is a Borel regular measure).

We denote by $\mathrm{M}(\mathbb{R}^n) = \mathrm{M}(\mathbb{R}^n, \mathbb{K}^m)$ the space of finite vector valued Radon measures.

From [9] we know that $\mathrm{M}(\mathbb{R}^n)$ equipped with the norm $\|\mu\|_{\mathrm{M}(\mathbb{R}^n)} := \|\mu\|_{\mathrm{M}} := |\mu|(\mathbb{R}^n)$ is a Banach space. Let $\mathscr{B}$ be the Borel σ-algebra. Since $\mathbb{K}^m$ has the Radon–Nikodým property, there exists a $\nu_\mu \in L^1(\mathbb{R}^n, |\mu|)$ such that we have $\mu(\mathcal{O}) = \int_{\mathcal{O}} \nu_\mu \, d|\mu|$ for $\mathcal{O} \in \mathscr{B}$. For $\psi \in \mathrm{BC}(\mathbb{R}^n, \mathbb{K}^{m\times\ell})$ we set

$$\mu\lfloor\psi(\mathcal{O}) := \int_{\mathcal{O}} \psi\nu_\mu \, d|\mu| \quad (\mathcal{O} \in \mathscr{B}),$$

which is well-defined since $\mathscr{B} \subset \mathscr{A}$. Elementary properties are listed in

Lemma 2.2. *Let* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $n, m, \ell, j \in \mathbb{N}$, $\phi \in \mathrm{BC}(\mathbb{R}^n, \mathbb{K}^{\ell\times j})$, *and* $\psi \in \mathrm{BC}(\mathbb{R}^n, \mathbb{K}^{m\times\ell})$. *Then we have* (1) $|\mu\lfloor\psi| \leqslant |\mu|\lfloor|\psi|$, (2) $\mu\lfloor\psi \in \mathrm{M}(\mathbb{R}^n)$, (3) $(\mu\lfloor\psi)\lfloor\phi = \mu\lfloor(\phi\psi)$.

Next, we consider the closed subspace of $\mathrm{M}(\mathbb{R}^n)$ consisting of measures with no point mass at the origin, i.e.,

$$\mathrm{M}_0(\mathbb{R}^n) := \{\mu \in \mathrm{M}(\mathbb{R}^n) : \mu(\{0\}) = 0\}.$$

We observe that

$$L^1(\mathbb{R}^n) \hookrightarrow \mathrm{M}_0(\mathbb{R}^n) \hookrightarrow \mathrm{M}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Hence, the Fourier transform of a Radon measure is defined and given as

$$\hat{\mu}(\xi) = \mu \lfloor \phi_\xi(\mathbb{R}^n) \quad \text{with } \phi_\xi(x) = (2\pi)^{-\frac{n}{2}} e^{-ix\cdot\xi}.$$

Spaces of Fourier transformed Radon measures then are defined as

$$\mathrm{FM}(\mathbb{R}^n) := \{\hat{\mu} : \mu \in \mathrm{M}(\mathbb{R}^n)\},$$
$$\mathrm{FM}_0(\mathbb{R}^n) := \{\hat{\mu} : \mu \in \mathrm{M}_0(\mathbb{R}^n)\},$$

which are equipped with the norm $\|u\|_{\mathrm{FM}} := \|\mathcal{F}^{-1}u\|_{\mathrm{M}} = \|\mathcal{F}u\|_{\mathrm{M}}$. Both $\mathrm{FM}(\mathbb{R}^n)$ and $\mathrm{FM}_0(\mathbb{R}^n)$ are Banach spaces. Furthermore, we define

$$\mathrm{FM}^k(\mathbb{R}^n) := \{u \in \mathrm{FM}(\mathbb{R}^n) : \partial^\alpha u \in \mathrm{FM}(\mathbb{R}^n)(|\alpha| \leqslant k)\}$$

for $k \in \mathbb{N}$ and

$$\mathrm{FM}^s(\mathbb{R}^n) := \{u \in \mathrm{FM}(\mathbb{R}^n) : (\xi \mapsto \hat{u}\lfloor|\xi|^s) \in \mathrm{M}(\mathbb{R}^n)\}$$

for $s \geqslant 0$. The spaces $\mathrm{FM}_0^k(\mathbb{R}^n)$ for $k \in \mathbb{N}$ and $\mathrm{FM}_0^s(\mathbb{R}^n)$ for $s \geqslant 0$ are defined accordingly. Note that for $s \in \mathbb{N}$ the two definitions are consistent thanks to Proposition 2.4 below. From [9] we recall the following useful facts.

Lemma 2.3. *Let $u, v \in \mathrm{FM}(\mathbb{R}^n)$. Then we have*

(1) $\|uv\|_{\mathrm{FM}} \leqslant (2\pi)^{-\frac{n}{2}} \|u\|_{\mathrm{FM}} \|v\|_{\mathrm{FM}}$,
(2) $\mathcal{F}L^1(\mathbb{R}^n) \hookrightarrow \mathrm{FM}_0(\mathbb{R}^n) \hookrightarrow \mathrm{FM}(\mathbb{R}^n) \hookrightarrow \mathrm{BUC}(\mathbb{R}^n)$.

Proposition 2.4. *For $\sigma \in \mathrm{BC}(\mathbb{R}^n\backslash\{0\}, \mathbb{K}^{m\times\ell})$ we set $Op(\sigma)f := \mathcal{F}^{-1}(\hat{f}\lfloor\sigma)$. Then we have*

$$\|Op(\sigma)\|_{\mathscr{L}(\mathrm{FM}_0(\mathbb{R}^n,\mathbb{K}^m),\mathrm{FM}_0(\mathbb{R}^n,\mathbb{K}^\ell))} = \|\sigma\|_{L^\infty(\mathbb{R}^n\backslash\{0\},\mathbb{K}^{m\times\ell})}.$$

If σ is additionally continuous at the origin, then the assertion holds also with FM_0 *replaced by* FM.

Remark 2.5. By the fact that $\mathcal{F}e^{ik\cdot} = (2\pi)^{\frac{n}{2}}\delta(\cdot - k)$ with δ the Dirac measure, we obtain $\|e^{ik\cdot}\|_{\mathrm{FM}} = (2\pi)^{\frac{n}{2}}\|\delta(\cdot - k)\|_{\mathrm{M}} < \infty$. Hence $e^{ik\cdot} \in \mathrm{FM}_0(\mathbb{R}^n)$ for $k \neq 0$ which proves the wave ansatz (2) to be a function in $\mathrm{FM}_0(\mathbb{R}^n)$.

3. Well-posedness, stability, and turbulence

We consider the following physically relevant stationary solutions:

$$(v, p) = (0, p_0) \tag{3}$$

with a pressure constant p_0 and, if $\alpha < 0$, additionally

$$(v, p) = (V, p_0), \tag{4}$$

where $V \in B_{\alpha,\beta} := \{x \in \mathbb{R}^n : |x| = \sqrt{-\alpha/\beta}\}$, i.e., V denotes a constant vector with arbitrary orientation and fixed swimming speed $|V| = \sqrt{-\alpha/\beta}$. The steady state (3) corresponds to a *disordered isotropic state* and (4) to the manifold $B_{\alpha,\beta}$ of *globally ordered polar states*.

In order to include the steady states, as in [26] we consider the following generalized system:

$$\begin{aligned} &u_t + \lambda_0\big[(u + V)\cdot\nabla\big]u + \big(M + \beta|u|^2\big)u - \Gamma_0\Delta u + \Gamma_2\Delta^2 u + \nabla q = f + N(u), \\ &\operatorname{div} u = 0, \\ &u(0) = u_0. \end{aligned} \tag{5}$$

Here $q = p - \lambda_1|v|^2$, $M \in \mathbb{R}^{n\times n}$ is a symmetric matrix, and $N(u) = \sum_{j,k} a_{jk}u^j u^k$ with $(a_{jk})_{j,k=1}^n \subset \mathbb{R}^{n\times n}$ is a quadratic nonlinear term. By setting

$$V = 0, \qquad M = \alpha I, \qquad N(u) = 0, \tag{6}$$

where I denotes the identity matrix and α is a scalar, we obtain (1) for $u = v$, i.e., the system corresponding to the steady state (3) and by setting

$$V \in B_{\alpha,\beta}, \qquad M = 2\beta V V^T, \qquad N(u) = -\beta|u|^2 V - 2\beta(u\cdot V)u \tag{7}$$

we obtain the system for $u = v - V$ corresponding to (4). Note that for the appearing parameters we always assume that

$$\lambda_0, \lambda_1, \Gamma_0, \alpha \in \mathbb{R}; \qquad \Gamma_2, \beta > 0. \tag{8}$$

Furthermore, space dimension is always assumed to be $n = 2$ or $n = 3$.

3.1. The linearized system

In this subsection we consider the linearized system

$$\begin{aligned} &u_t + \lambda_0(V\cdot\nabla)u + Mu - \Gamma_0\Delta u + \Gamma_2\Delta^2 u + \nabla q = f \quad \text{in } (0,\infty)\times\mathbb{R}^n, \\ &\operatorname{div} u = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^n, \\ &u(0) = u_0 \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{9}$$

In a first step we introduce the Helmholtz projection on $\mathrm{FM}_0(\mathbb{R}^n)$. The symbol of the Helmholtz projection is defined as $\sigma_P(\xi) := I - \xi\xi^T/|\xi|^2$ and the corresponding operator as $Pu := \mathcal{F}^{-1}(\hat{u}\lfloor\sigma_P)$ for $u \in \mathrm{FM}_0(\mathbb{R}^n)$. Note that P is bounded on $\mathrm{FM}_0(\mathbb{R}^n)$ by Proposition 2.4. We obtain the Helmholtz decomposition

$$\mathrm{FM}_0\big(\mathbb{R}^n\big) = \mathrm{FM}_{0,\sigma}\big(\mathbb{R}^n\big) \oplus G_{\mathrm{FM}}\big(\mathbb{R}^n\big),$$

with

$$\begin{aligned} \mathrm{FM}_{0,\sigma}\big(\mathbb{R}^n\big) &:= P\mathrm{FM}_0\big(\mathbb{R}^n\big) = \big\{u \in \mathrm{FM}_0\big(\mathbb{R}^n\big) : \operatorname{div} u = 0\big\}, \\ G_{\mathrm{FM}}\big(\mathbb{R}^n\big) &:= \big\{\nabla p : p \in \widehat{\mathrm{FM}}{}^1_0\big(\mathbb{R}^n\big)\big\}, \end{aligned}$$

where $\widehat{\mathrm{FM}}{}^1_0(\mathbb{R}^n) = \{p \in \mathcal{D}'(\mathbb{R}^n) : \nabla p \in \mathrm{FM}_0(\mathbb{R}^n)\}/\mathbb{C}$, see [9]. Next, we define the operator associated to (9) as

$$\begin{aligned} A_{\mathrm{LF}}u &:= \lambda_0(V\cdot\nabla)u + PMu - \Gamma_0\Delta u + \Gamma_2\Delta^2 u, \\ D(A_{\mathrm{LF}}) &:= \mathrm{FM}^4_{0,\sigma}\big(\mathbb{R}^n\big) := \mathrm{FM}_{0,\sigma}\big(\mathbb{R}^n\big) \cap \mathrm{FM}^4_0\big(\mathbb{R}^n\big). \end{aligned} \tag{10}$$

The Fourier symbol of the operator A_{LF} then reads as

$$\sigma_{A_{\mathrm{LF}}}(\xi) := \mathcal{F}^{-1}A_{\mathrm{LF}}\mathcal{F} = \Gamma_2|\xi|^4 + \Gamma_0|\xi|^2 + \sigma_P(\xi)M + i\lambda_0 V\cdot\xi, \quad \xi \in \mathbb{R}^n.$$

Again thanks to Proposition 2.4 we can derive a bounded H^∞-calculus. For an introduction to the notion of a bounded H^∞-calculus and its use we refer to [2,12,18]. To be precise, we have

Proposition 3.1. *There exists an $\omega > 0$ such that $\omega + A_{\mathrm{LF}}$ admits a bounded H^∞-calculus on* $\mathrm{FM}_{0,\sigma}(\mathbb{R}^n)$ *with H^∞-angle $\phi^\infty_{\omega+A_{\mathrm{LF}}} < \pi/2$.*

Proof. Since $\Gamma_2 > 0$ there exists an $\omega > 0$ and a $\varphi_0 \in (0, \pi/2)$ such that $\omega + \sigma_{A_{\mathrm{LF}}} \in \overline{\Sigma_{\varphi_0}}$ and $|\omega + \sigma_{A_{\mathrm{LF}}}| \geqslant \delta > 0$ on $\mathbb{R}^n \setminus \{0\}$. Thus for $\varphi \in (\varphi_0, \pi/2)$ the symbol $\xi \mapsto h(\omega + \sigma_{A_{\mathrm{LF}}}(\xi))\sigma_P(\xi)$ is bounded and continuous on $\mathbb{R}^n \setminus \{0\}$ and satisfies

$$\big\|h(\omega + \sigma_{A_{\mathrm{LF}}})\sigma_P\big\|_{L^\infty(\mathbb{R}^n)} \leqslant C_{\varphi_0}\|h\|_\infty \quad \big(h \in H^\infty(\Sigma_\varphi)\big),$$

where $\Sigma_\varphi := \{z \in \mathbb{C} \setminus \{0\}; |\arg z| < \varphi\}$ and $H^\infty(\Sigma_\varphi)$ denotes the space of bounded holomorphic functions on the sector Σ_φ. By the fact that

$$h(\omega + A_{\mathrm{LF}})Pu = \mathcal{F}^{-1}\big(\hat{u}\lfloor h(\omega + \sigma_{A_{\mathrm{LF}}})\sigma_P\big) \quad \big(u \in \mathrm{FM}_0\big(\mathbb{R}^n\big)\big)$$

Proposition 2.4 yields

$$\big\|h(\omega + A_{\mathrm{LF}})P\big\|_{\mathscr{L}(\mathrm{FM}_0(\mathbb{R}^n))} \leqslant C_{\varphi_0}\|h\|_\infty \quad \big(h \in H^\infty(\Sigma_\varphi)\big). \tag{11}$$

Setting $h(z) := \lambda(\lambda + z)^{-1}$, estimate (11) and the fact that $\omega + A_{\mathrm{LF}}$ is invertible imply sectoriality of $\omega + A_{\mathrm{LF}}$ on $\mathrm{FM}_{0,\sigma}(\mathbb{R}^n)$ with spectral angle $\phi_{\omega+A_{\mathrm{LF}}} < \pi/2$. Thus the holomorphic functional calculus via the Dunford integral is defined as usual, see [2]. Estimate (11) then yields the assertion. □

Note that by the sectoriality of $\omega + A_{LF}$ the operator $-A_{LF}$ generates an analytic C_0-semigroup on $FM_{0,\sigma}(\mathbb{R}^n)$. Furthermore, fractional powers $(\omega + A_{LF})^\gamma : D((\omega + A_{LF})^\gamma) \to FM_{0,\sigma}(\mathbb{R}^n)$, $\gamma > 0$, are well-defined, see [2]. As a consequence of Proposition 3.1 we immediately obtain

Corollary 3.2. *For $\gamma \in (0, 1)$ we have*

$$[FM_{0,\sigma}(\mathbb{R}^n), D(A_{LF})]_\gamma = D\big((\omega + A_{LF})^\gamma\big) = FM_0^{4\gamma}(\mathbb{R}^n) \cap FM_{0,\sigma}(\mathbb{R}^n),$$

where $[\cdot, \cdot]_\gamma$ denotes the complex interpolation functor.

Proof. By means of Fourier transformation it is straight forward to verify the second equality, whereas the first equality is a consequence of [23, Theorem 1.15.3]. □

One advantage of working in $FM(\mathbb{R}^n)$ is reflected by the fact that the operator $\Gamma_2\Delta^2$ with domain $D(\Gamma_2\Delta^2) = FM^4(\mathbb{R}^n)$ has L^1 maximal regularity.

Proposition 3.3. *Let $1 \leqslant p \leqslant \infty$. For $T(t) := \exp(-\Gamma_2 t\Delta^2)$ and $(\Delta^2 T \star f)(t) := \Delta^2 \int_0^t T(t - s) f(s)\, ds$ we have*

(1) $\|\Delta^2 T u_0\|_{L^p(\mathbb{R}_+, FM(\mathbb{R}^n))} \leqslant \frac{1}{(\Gamma_2 p)^{1/p}} \|u_0\|_{FM^{4-4/p}}$,
(2) $\|\Delta^2 T \star f\|_{L^1(\mathbb{R}_+, FM(\mathbb{R}^n))} \leqslant \frac{1}{\Gamma_2} \|f\|_{L^1(\mathbb{R}_+, FM(\mathbb{R}^n))}$.

Proof. To prove (1) we have due to Lemma 2.2(1) that

$$\big\|\Delta^2 e^{-\Gamma_2 t\Delta^2} u_0\big\|_{FM} = \big\|\widehat{u}_0 \llcorner \big(|\xi|^4 e^{-\Gamma_2 t|\xi|^4}\big)\big\|_M \leqslant \int_{\mathbb{R}^n} |\xi|^4 \big|e^{-\Gamma_2 t|\xi|^4}\big|\, d|\widehat{u}_0|(\xi).$$

Then the assertion follows since

$$\begin{aligned}\big\|\Delta^2 T u_0\big\|_{L^p(\mathbb{R}_+, FM(\mathbb{R}^n))} &\leqslant \int_{\mathbb{R}^n} |\xi|^4 \big\|e^{-\Gamma_2 (\cdot)|\xi|^4}\big\|_{L^p(\mathbb{R}_+)}\, d|\widehat{u}_0|(\xi) \\ &\leqslant \frac{1}{(\Gamma_2 p)^{1/p}} \|u_0\|_{FM^{4-4/p}}.\end{aligned}$$

Estimate (2) follows from (1) and [8, Lemma 2.4]. □

Consequently, A_{LF} has L^1 maximal regularity as well:

Theorem 3.4. *Let $T \in (0, \infty)$. For $f \in L^1((0, T), FM_{0,\sigma}(\mathbb{R}^n))$ and $u_0 \in FM_{0,\sigma}(\mathbb{R}^n)$ there exists a unique solution (u, q) of (9) satisfying*

$$\begin{aligned}&\|u\|_{W^{1,1}((0,T),FM)} + \|u\|_{L^1((0,T),FM^4)} + \|\nabla q\|_{L^1((0,T),FM)} \\ &\quad \leqslant C(T)\big(\|f\|_{L^1((0,T),FM)} + \|u_0\|_{FM}\big)\end{aligned}$$

with $C(T) > 0$ independent of u, q, f, u_0.

Proof. By Proposition 3.3 the operator $\Gamma_2\Delta^2$ enjoys L^1 maximal regularity also on $\mathrm{FM}_{0,\sigma}(\mathbb{R}^n)$. Since the remaining terms in A_{LF} are of lower order, the assertion follows by a standard perturbation argument. □

Now we consider the spectrum of A_{LF} in order to examine stability. For this purpose we set $A_d := A_{\mathrm{LF}}$ in case of the disordered state (3), that is, $V = 0$ and $M = \alpha I$ and thus $\sigma_P(\xi)M$ is simply replaced by α since A_d acts on the solenoidal vector fields. Then the Fourier symbol of A_d is given as

$$\sigma_{A_d}(\xi) := \Gamma_2|\xi|^4 + \Gamma_0|\xi|^2 + \alpha, \quad \xi \in \mathbb{R}^n.$$

If we substitute $s = |\xi|^2$ we can characterize the spectrum of $-A_d$ easily by computing the intersection points of σ_{A_d}. We obtain

$$s_\pm = -\frac{\Gamma_0}{\Gamma_2}\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\alpha\Gamma_2}{\Gamma_0^2}}\right) \tag{12}$$

and the following result on (in-)stability:

Proposition 3.5. *Assume (8). Then the C_0-semigroup $(\exp(-tA_d))_{t\geqslant 0}$ on $\mathrm{FM}_{0,\sigma}(\mathbb{R}^n)$, which corresponds to the disordered isotropic state (3), is linearly stable if $\Gamma_0 < 0$ and $4\alpha > \Gamma_0^2/\Gamma_2$ or if $\Gamma_0 \geqslant 0$ and $\alpha > 0$. More precisely, it is*

(1) *exponentially stable if $\Gamma_0 < 0$ and $4\alpha > \Gamma_0^2/\Gamma_2$ or if $\Gamma_0 \geqslant 0$ and $\alpha > 0$;*
(2) *asymptotically stable if $\Gamma_0 < 0$ and $4\alpha = \Gamma_0^2/\Gamma_2$ or if $\Gamma_0 \geqslant 0$ and $\alpha = 0$;*
(3) *exponentially unstable if $\Gamma_0 < 0$ and $4\alpha < \Gamma_0^2/\Gamma_2$ or if $\Gamma_0 \geqslant 0$ and $\alpha < 0$.*

Proof. For the exponential (in-)stability we note that the growth bound $\omega((\exp(-tA_d))_{t\geqslant 0})$ and the spectral bound $s(-A_d) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(-A_d)\}$ coincide, since $(\exp(-tA_d))_{t\geqslant 0}$ is an analytic C_0-semigroup, see [19]. Thanks to (12), relations (1) and (3) are immediate. In case of (2) we obtain by Lemma 2.2(1) that

$$\left\|\exp(-tA_d)u_0\right\|_{\mathrm{FM}} \leqslant |\widehat{u_0}|\llcorner\left|e^{-t\sigma_{A_d}}\right|\left(\mathbb{R}^n\right) = \int_{\mathbb{R}^n}\left|e^{-t\sigma_{A_d}}\right| d|\widehat{u_0}|.$$

Dominated convergence implies $\exp(-tA_d)u_0 \to 0$ for $t \to \infty$ and the assertion is proved. □

Next, we consider the ordered polar state (4). We set $A_o := A_{\mathrm{LF}}$ in this case and

$$\sigma_{A_o}(\xi) := \Gamma_2|\xi|^4 + \Gamma_0|\xi|^2 + i\lambda_0(V\cdot\xi) + 2\beta\sigma_P(\xi)VV^T, \quad \xi \in \mathbb{R}^n.$$

We note that $\sigma_P(\xi)VV^T$ is a positive semidefinite matrix and zero is an eigenvalue with eigenvector $x \in \{V\}^\perp$. Choosing $x, \xi \in \{V\}^\perp$ with $|x| = 1$ and $|\xi|$ sufficiently small, we can achieve that

$$x^T\sigma_{A_o}(\xi)x = \Gamma_2|\xi|^4 + \Gamma_0|\xi|^2 < 0,$$

if $\Gamma_0 < 0$. This proves

Proposition 3.6. *Assume (8). Then the C_0-semigroup $(\exp(-tA_o))_{t\geqslant 0}$ corresponding to the ordered polar state (4) is*

(1) *exponentially unstable on* $\mathrm{FM}_{0,\sigma}(\mathbb{R}^n)$ *if* $\Gamma_0 < 0$*;*
(2) *asymptotically stable on* $\mathrm{FM}_{0,\sigma}(\mathbb{R}^n)$ *if* $\Gamma_0 \geqslant 0$.

Proof. Assertion (1) is clear due to the discussion above. Assertion (2) follows completely analogous to the proof of Proposition 3.5(2). □

3.2. Local strong and global mild solvability

We first construct a maximal solution which includes local well-posedness. For $T > 0$ we define relevant function spaces as

$$\begin{aligned}
\mathbb{E}_T &:= W^{1,1}\big((0,T),\mathrm{FM}_{0,\sigma}(\mathbb{R}^n)\big) \cap L^1\big((0,T),\mathrm{FM}_0^4(\mathbb{R}^n)\big),\\
{}_0\mathbb{E}_T &:= {}_0W^{1,1}\big((0,T),\mathrm{FM}_{0,\sigma}(\mathbb{R}^n)\big) \cap L^1\big((0,T),\mathrm{FM}_0^4(\mathbb{R}^n)\big),\\
\mathbb{F}_T^1 &:= L^1\big((0,T),\mathrm{FM}_{0,\sigma}(\mathbb{R}^n)\big), \qquad \mathbb{F}^2 := \mathrm{FM}_{0,\sigma}(\mathbb{R}^n),\\
\mathbb{F}_T &:= \mathbb{F}_T^1 \times \mathbb{F}^2,
\end{aligned}$$

and the linear operator

$$L : \mathbb{E}_T \to \mathbb{F}_T, \qquad Lu := \big(\partial_t u + A_{\mathrm{LF}}u, u(0)\big).$$

Here $u \in {}_0W^{1,1}$ means that $u|_{t=0} = 0$. If we also set

$$H(u) := \beta P|u|^2u + \lambda_0 P(u\cdot\nabla)u - PN(u), \tag{13}$$

$$F(u) := Lu + \big(H(u), 0\big), \tag{14}$$

then the full system (5) is rephrased as $F(u) = (f, u_0)$.

Lemma 3.7. *We have* $H \in C^1(\mathbb{E}_T, \mathbb{F}_T^1)$ *and its Fréchet derivative is represented as*

$$DH(v)u = P\sum_{|\alpha|\leqslant 1} b_\alpha \partial^\alpha u + \lambda_0 P(u\cdot\nabla)v, \quad u, v \in \mathbb{E}_T, \tag{15}$$

with matrices $b_\alpha = b_\alpha(v) \in L^\infty((0,T),\mathrm{FM}_0(\mathbb{R}^n,\mathbb{C}^{n\times n}))$.

Proof. First observe that the Sobolev embedding

$$W^{1,1}\big((0,T),X\big) \hookrightarrow \mathrm{BUC}\big((0,T),X\big) \tag{16}$$

yields

$$\mathbb{E}_T \hookrightarrow W^{1,1}\big((0,T),\mathrm{FM}_0(\mathbb{R}^n)\big) \hookrightarrow \mathrm{BUC}\big((0,T),\mathrm{FM}_0(\mathbb{R}^n)\big). \tag{17}$$

Utilizing this and the algebra property of FM_0, we easily obtain

$$\begin{aligned}
&\big\|(u\cdot\nabla)u\big\|_{\mathbb{F}_T^1}\leqslant C\|u\|_{L^\infty(\mathrm{FM}_0)}\|\nabla u\|_{\mathbb{F}_T^1}\leqslant C\|u\|_{\mathbb{E}_T}^2,\\
&\big\||u|^2u\big\|_{\mathbb{F}_T^1}\leqslant C\|u\|_{L^\infty(\mathrm{FM}_0)}^2\|u\|_{\mathbb{F}_T^1}\leqslant C\|u\|_{\mathbb{E}_T}^3,\\
&\big\|N(u)\big\|_{\mathbb{F}_T^1}\leqslant C\|u\|_{L^\infty(\mathrm{FM}_0)}\|u\|_{\mathbb{F}_T^1}\leqslant C\|u\|_{\mathbb{E}_T}^2,
\end{aligned}$$

hence $H:\mathbb{E}_T\to\mathbb{F}_T^1$. By the fact that H consists of bi- and trilinear terms, it is obvious that $H\in C^1(\mathbb{E}_T,\mathbb{F}_T^1)$ (even $H\in C^\infty(\mathbb{E}_T,\mathbb{F}_T^1)$). The Fréchet derivative reads as

$$DH(v)u=\beta P|v|^2u+2\beta P(u\cdot v)v+\lambda_0P(u\cdot\nabla)v+\lambda_0P(v\cdot\nabla)u-2P\sum_{j,k=1}^n a_{jk}\big(u^jv^k+u^kv^j\big).$$

From this and (17) representation (15) obviously follows. □

Lemma 3.8. *Let* $T\in(0,\infty)$ *and fix* $v\in\mathbb{E}_T$. *Then we have*

$$L+\big(DH(v),0\big)\in\mathscr{L}_{\mathrm{is}}(\mathbb{E}_T,\mathbb{F}_T).$$

Proof. By employing representation (15) for $B(t):=(DH(v(t)),0)$ we will show that $B(\cdot)$ is a suitable perturbation of L. In the proof we avoid the use of mixed derivative type theorems, since their availability in the underlying situation is not clear. Therefore we proceed in two steps.

First we will show that $B_1(t)u:=P\sum_{|\alpha|\leqslant1}b_\alpha(t)\partial^\alpha u$ is relatively bounded by $A_{\mathrm{LF}}+\mu$ for $\mu>0$ large enough. Utilizing the algebra property of FM_0 we can estimate

$$\begin{aligned}
&\big\|B_1(t)u\big\|_{\mathrm{FM}_0}\\
&\quad\leqslant C\big(\big\||v(t)|^2\big\|_{\mathrm{FM}_0}+\big\|v(t)\big\|_{\mathrm{FM}_0}\big)\|u\|_{\mathrm{FM}_0^1}\\
&\quad\leqslant\frac{C}{\mu^{3/4}}\big(\|v\|_{L^\infty((0,T),\mathrm{FM}_0)}^2+\|v\|_{L^\infty((0,T),\mathrm{FM}_0)}\big)\big\|(\mu+A_{\mathrm{LF}})u\big\|_{\mathrm{FM}_0}
\end{aligned}$$

for all $t\in(0,T)$, $u\in D(A_{\mathrm{LF}})$ and $\mu\geqslant\mu_0$ with a certain $\mu_0>0$. Thus, choosing μ large enough we can apply [21, Theorem 2.5] to the result that

$$L+(\mu+B_1,0)\in\mathscr{L}_{\mathrm{is}}(\mathbb{E}_T,\mathbb{F}_T).$$

Since $L+(\mu+B_1,0)$ is linear, we can remove the shift $\mu>0$. (Note that in [21, Theorem 2.5] it is assumed that $p>1$. With the Definition of L^1 maximal regularity used here it is obvious, however, that the Theorem remains true for $p=1$.)

In the second step we show that $B_2u:=(\lambda_0P(u\cdot\nabla)v,0)$ is a lower order perturbation of $L+(B_1,0)$. To this end, we first consider the case of zero time trace, that is, $u\in{}_0\mathbb{E}_T$. Observe that then the embedding constant in the Sobolev embedding (16) does not depend on the length of the interval $(0,T)$, if we replace $W^{1,1}$ by its zero trace version ${}_0W^{1,1}$. As a consequence embedding (17) is independent of T, too. This yields

$$\begin{aligned}\big\|(u\cdot\nabla)v\big\|_{L^1((0,T'),\mathrm{FM}_0)} &\leqslant C\|\nabla v\|_{L^1((0,T'),\mathrm{FM}_0)}\|u\|_{L^\infty((0,T'),\mathrm{FM}_0)}\\ &\leqslant C\|\nabla v\|_{L^1((0,T'),\mathrm{FM}_0)}\|u\|_{{}_0\mathbb{E}_{T'}}\quad \big(T'\in(0,T)\big),\end{aligned}$$

and we obtain

$$\|B_2u\|_{\mathbb{F}_{T'}}\leqslant C\|\nabla v\|_{L^1((0,T'),\mathrm{FM}_0)}\|u\|_{{}_0\mathbb{E}_{T'}}$$

for all $T'\in(0,T)$ and $u\in{}_0\mathbb{E}_{T'}$. Thus, choosing $T'\in(0,T)$ small enough, a standard Neumann series argument implies

$$L+B\in\mathscr{L}_{\mathrm{is}}({}_0\mathbb{E}_{T'},\mathbb{F}_{T'}). \tag{18}$$

Since $L+B$ is linear and $\|v\|_{\mathbb{E}_T}<\infty$, we can iterate this procedure. Consequently, (18) remains true for $T'=T$. This implies that $A_{\mathrm{LF}}+DH(v)$ has maximal regularity on $\mathrm{FM}_{0,\sigma}$. Thus (18) remains valid for general time trace in $\mathbb{F}^2$. $\square$

Appealing to the local inverse theorem, we can now prove the following result.

Proposition 3.9 (Maximal solution)**.** *Assume that (8) holds. For every $f\in L^1((0,\infty),\mathrm{FM}_{0,\sigma}(\mathbb{R}^n))$ and $u_0\in\mathrm{FM}_{0,\sigma}(\mathbb{R}^n)$ there exists a $T^*>0$ and a unique maximal strong solution (u,q) of (5) such that*

$$u\in\mathbb{E}_T,\qquad \nabla q\in L^1\big((0,T),\mathrm{FM}_0\big(\mathbb{R}^n\big)\big)$$

for all $T\in(0,T^*)$. Either we have $T^*=\infty$ or the maximal solution satisfies

$$\limsup_{t\to T^*}\big\|u(t)\big\|_{\mathrm{FM}_0}=\infty.$$

Proof. We fix $(f,u_0)\in\mathbb{F}_T$ and define a reference solution as

$$u^*:=L^{-1}(f,u_0)\in\mathbb{E}_T.$$

For the Fréchet derivative of the nonlinear operator $F\in C^1(\mathbb{E}_T,\mathbb{F}_T)$ given in (14) we obtain in view of Lemma 3.8 that

$$DF\big(u^*\big)=L+\big(DH\big(u^*\big),0\big)\in\mathscr{L}_{\mathrm{is}}(\mathbb{E}_T,\mathbb{F}_T).$$

Utilizing the local inverse theorem, the construction of a unique local strong solution follows now verbatim the lines of the proof of [26, Theorem 1].

Based on the local well-posedness, as usual, we can show the existence of a $T^*>0$ and of a unique non-extendible solution (u,q) on $(0,T^*)$. For the additional property, suppose that

$\limsup_{t\to T^*}\|u(t)\|_{\mathrm{FM}_0} < \infty$ and nevertheless $T^* < \infty$. This implies $u \in \mathrm{BC}([0,T^*),\mathrm{FM}_0)$ thanks to $u \in \mathbb{E}_T$ for $T < T^*$ and embedding (17). Next, we write

$$H(u)(t) = \big(\beta P\big|u(t)\big|^2 + \lambda_0 P\big(u(t)\cdot\nabla\big) - Pu(t)^T A\big)u(t) =: B(t)u$$

with $A = (a_{jk})_{j,k=1}^n$. This allows for regarding (5) as the 'linear' system

$$\big(\partial_t u + A_{\mathrm{LF}}u + B(\cdot)u, u(0)\big) = (f, u_0).$$

By the fact that

$$\big\|B(t)u\big\|_{\mathrm{FM}_0} \leqslant C\big\|u(t)\big\|_{\mathrm{FM}_0^1} \quad \big(t \in \big(0, T^*\big)\big),$$

we see that $B(t)$ is a lower order perturbation. It is well-known that then maximal regularity remains true for $A_{\mathrm{LF}} + B(\cdot)$. In fact, based on a Neumann series argument very similar as, e.g., in [21, Theorem 2.5] or [26, Lemma 3] it can be proved that

$$L + \big(B(\cdot), 0\big) \in \mathscr{L}_{\mathrm{is}}(\mathbb{E}_{T^*}, \mathbb{F}_{T^*}).$$

By the uniqueness of the solution and due to (17) this gives us

$$u \in \mathbb{E}_{T^*} \hookrightarrow \mathrm{BUC}\big(\big(0, T^*\big), \mathrm{FM}_{0,\sigma}\big).$$

Thus $\lim_{t\to T^*}\|u(t)\|_{\mathrm{FM}_0}$ exists and starting from the initial value $u(T^*)$ we can extend the solution u beyond T^* which contradicts its non-extendability. □

In the case of linear exponential stability we obtain existence of a global mild solution for small data, i.e., a solution of the variation of constant formula

$$u(t) = \exp(-tA_d)u_0 + \int_0^t \exp\big(-(t-s)A_d\big)H(u)(s)\,ds, \quad t > 0. \tag{19}$$

Besides, the exponential stability transfers to the nonlinear system.

Theorem 3.10. *Assume (8) such that $\Gamma_0 < 0$ and $4\alpha > \Gamma_0^2/\Gamma_2$, or such that $\Gamma_0 \geqslant 0$ and $\alpha > 0$. Then there is a $\kappa > 0$ such that, if $\|u_0\|_{\mathrm{FM}} < \kappa$, there exists a unique global mild solution $u \in \mathrm{BC}([0,\infty), \mathrm{FM}_{0,\sigma}(\mathbb{R}^n))$ of (19) satisfying*

$$\big\|u(t)\big\|_{\mathrm{FM}} \leqslant C\mathrm{e}^{-\omega t}\|u_0\|_{\mathrm{FM}} \quad (t \geqslant 0)$$

for some $C, \omega > 0$. Furthermore, recovering the pressure via

$$\nabla q := -(I-P)\big[\lambda_0[u\cdot\nabla]u + \big(M + \beta|u|^2\big)u - N(u)\big] \in L^1\big((0,T), \mathrm{FM}_0\big(\mathbb{R}^n\big)\big)$$

the pair $(u, \nabla q)$ is the unique classical solution of (5).

Proof. The proof is very analogous to the proof of [7, Theorems 1.2 and 1.3] and is hence omitted. □

3.3. Nonlinear turbulence

Most of the outcome on linear (in-) stability in the FM-setting transfers to the corresponding nonlinear situation. The transfer of turbulence follows by principles on linearized instability. Here we apply [14, Corollary 5.1.6].

Lemma 3.11. *Consider the nonlinearity H given in (13). Then we have $H \in C^1(\mathrm{FM}^\eta(\mathbb{R}^n), \mathrm{FM}_{0,\sigma}(\mathbb{R}^n))$ for $\eta \geqslant 1$ and the estimate*

$$\|H(u)\|_{\mathrm{FM}} \leqslant C\|u\|_{\mathrm{FM}^\eta}^2 \quad \left(\|u\|_{\mathrm{FM}^\eta} \leqslant 1\right).$$

Proof. Using the algebra property of $\mathrm{FM}_0(\mathbb{R}^n)$ in Lemma 2.3(1) we obtain

$$\begin{aligned}
&\|(u\cdot\nabla)u\|_{\mathrm{FM}} \leqslant C\|u\|_{\mathrm{FM}}\|\nabla u\|_{\mathrm{FM}} \leqslant C\|u\|_{\mathrm{FM}}\|u\|_{\mathrm{FM}^\eta},\\
&\||u|^2u\|_{\mathrm{FM}} \leqslant C\|u\|_{\mathrm{FM}}^3,\\
&\|N(u)\|_{\mathrm{FM}} \leqslant C\|u\|_{\mathrm{FM}}^2,
\end{aligned}$$

and the claimed estimate follows for $u \in \mathrm{FM}^\eta(\mathbb{R}^n)$ with $\|u\|_{\mathrm{FM}^\eta} \leqslant 1$. The estimates also prove $H \in C^1(\mathrm{FM}^\eta(\mathbb{R}^n), \mathrm{FM}_{0,\sigma}(\mathbb{R}^n))$, since H consists of bi- and trilinear terms. □

First, we again examine the (in-)stability of the disordered state (3).

Theorem 3.12. *Assume (8). Then the disordered state (3) is nonlinearly*

(1) *exponentially stable in $\mathrm{FM}_{0,\sigma}(\mathbb{R}^n)$ if $\Gamma_0 < 0$ and $4\alpha > \Gamma_0^2/\Gamma_2$, or if $\Gamma_0 \geqslant 0$ and $\alpha > 0$;*
(2) *unstable in $\mathrm{FM}^{4\gamma}(\mathbb{R}^n) \cap \mathrm{FM}_{0,\sigma}(\mathbb{R}^n)$ for $\gamma \in [1/4, 1)$ if $\Gamma_0 < 0$ and $4\alpha < \Gamma_0^2/\Gamma_2$, or if $\Gamma_0 \geqslant 0$ and $\alpha < 0$.*

Proof. (1) is an immediate consequence of Theorem 3.10.

For (2) first observe that $T^* < \infty$ implies that $u \equiv 0$ is unstable, since $\limsup_{t\to\infty} \|u(t)\|_{\mathrm{FM}_0} = \infty$ by Proposition 3.9. So, w.l.o.g. we can assume $T^* = \infty$. From Proposition 3.5 we readily have that $\sigma(-A_d) \cap \{z \in \mathbb{C} : \mathrm{Re} z > 0\} \neq \emptyset$. Thanks to Corollary 3.2 and Lemma 3.11 with $\eta = 4\gamma \geqslant 1$ we can apply [14, Corollary 5.1.6] and the assertion follows. (In the notation of [14] we have $x_0 = 0$, $A = A_d$, $B = 0$, $f(u) = g(u) = H(u)$, $\alpha = \gamma$, $p = 2$.) □

We obtain a similar result on instability of the ordered polar state (4).

Theorem 3.13. *Let $\Gamma_2 > 0$, $\beta > 0$ and $\Gamma_0, \alpha < 0$. Then the ordered polar state (4) is nonlinearly unstable in $\mathrm{FM}^{4\gamma}(\mathbb{R}^n) \cap \mathrm{FM}_{0,\sigma}(\mathbb{R}^n)$ for $\gamma \in [1/4, 1)$.*

Proof. Also here the assumptions of [14, Corollary 5.1.6] are fulfilled thanks to Corollary 3.2, Proposition 3.6(1) and Lemma 3.11. □

4. Conclusion

We gave an analytical approach to the active fluid model proposed by Wensink et al. [24] in the $\mathrm{FM}(\mathbb{R}^n)$-setting, i.e., in spaces of Fourier transformed Radon measures. In detail we have proved:

(i) existence of a unique maximal strong solution for arbitrary data and existence of a unique global mild (classical) solution for small data in case of linear exponential stability;
(ii) results on linear and nonlinear stability and instability of the ordered and the disordered steady states in the $\mathrm{FM}(\mathbb{R}^n)$-setting, depending on the values of the occuring physically relevant parameters.

By the fact that wave modes belong to $\mathrm{FM}(\mathbb{R}^n)$ (not to $L^2(\mathbb{R}^n)$) this justifies the typical formal stability analysis based on wave modes [24]. It also justifies mathematically the asymptotic behavior observed in simulations and experiments [1,3,11,15,24,25], in particular meso-scale turbulence caused by self-propulsion.

References

[1] K. Beppu, Z. Izri, J. Gohya, K. Eto, M. Ichikawa and Y.T. Maeda, Geometry-driven collective ordering of bacterial vortices, *Soft Matter* **13**(29) (2017), 5038–5043. doi:10.1039/C7SM00999B.
[2] R. Denk, M. Hieber and J. Prüß, $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, *Mem. Amer. Math. Soc.* (2003), 114–166.
[3] A. Doostmohammadi, T.N. Shendruk, K. Thijssen and J.M. Yeomans, Onset of meso-scale turbulence in active nematics, *Nat. Comm.* **8** (2017), 15326. doi:10.1038/ncomms15326.
[4] J. Dunkel, S. Heidenreich, M. Bär and R.E. Goldstein, Minimal continuum theories of structure formation in dense active fluids, *New Journal of Physics* **15** (2013), 045016. doi:10.1088/1367-2630/15/4/045016.
[5] J. Dunkel, S. Heidenreich, K. Drescher, H.H. Wensink, M. Bär and R.E. Goldstein, Fluid dynamics of bacterial turbulence, *Phys. Rev. Lett.* **110** (2013), 228102. doi:10.1103/PhysRevLett.110.228102.
[6] A. Fischer and J. Saal, On instability of the Ekman spiral, *Discrete Contin. Dyn. Syst. Ser* **6**(5) (2013), 1225–1236. doi:10.3934/dcdss.2013.6.1225.
[7] Y. Giga, K. Inui, A. Mahalov and J. Saal, Global solvabiliy of the Navier–Stokes equations in spaces based on sum-closed frequency sets, *Adv. Differ. Equ.* **12**(7) (2007), 721–736.
[8] Y. Giga and J. Saal, L^1 maximal regularity for the Laplacian and applications, *Discrete Contin. Dyn. Syst.* **2011**(Suppl) (2011), 495–504.
[9] Y. Giga and J. Saal, An approach to rotating boundary layers based on vector Radon measures, *J. Math. Fluid Mech.* **15**(1) (2013), 89–127. doi:10.1007/s00021-012-0094-1.
[10] Y. Giga and J. Saal, Uniform exponential stability of the Ekman spiral, *Ark. Mat.* **53** (2015), 105–126. doi:10.1007/s11512-014-0203-x.
[11] P. Guillamat, J. Ignes-Mullol and F. Sagues, Taming active turbulence with patterned soft interfaces, *Nat. Comm.* **8** (2017), 564. doi:10.1038/s41467-017-00617-1.
[12] M. Haase, *The Functional Calculus for Sectorial Operators*, Birkhäuser, 2006.
[13] S. Heidenreich, J. Dunkel, S.H.L. Klapp and M. Bär, Hydrodynamic length-scale selection in microswimmer suspensions, *Physical Review E* **94**(2) (2016), 020601. doi:10.1103/PhysRevE.94.020601.
[14] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer, Berlin-New York, 1981.
[15] A. Kaiser, A. Peshkow, A. Sokolov, B. ten Hagen, H. Löwen and I.S. Aranson, Transport powered by bacterial turbulence, *Physical Review Letters* **112** (2014), 158101. doi:10.1103/PhysRevLett.112.158101.
[16] A. Kaiser, A. Sokolov, I.S. Aranson and H. Löwen, Mechanisms of carrier transport induced by a microswimmer bath, *IEEE Transactions on Nanobioscience* **14** (2015), 260–266. doi:10.1109/TNB.2014.2361652.
[17] A. Kaiser, A. Sokolov, I.S. Aranson and H. Löwen, Motion of two micro-wedges in a turbulent bacterial bath, *European Physical Journal: Special Topics* **224** (2015), 1275–1286.
[18] P.C. Kunstmann and L. Weis, Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus, in: *Functional Analytic Methods for Evolution Equations*, M. Iannelli, R. Nagel and S. Piazzera, eds, Springer, Berlin Heidelberg, 2004, pp. 65–311. doi:10.1007/978-3-540-44653-8_2.

[19] R. Nagel, *One-Parameter Semigroups of Positive Operators*, Lecture Notes in Mathematics, Vol. 1184, Springer Verlag, Berlin, 1986.

[20] A. Oza and S. Heidenreich, Generalized Swift–Hohenberg models for dense active suspensions, *European Physical Journal E* **39** (2016), 97. doi:10.1140/epje/i2016-16097-2.

[21] J. Saal, Maximal regularity for the Stokes equations in non-cylindrical space-time domains, *J. Math. Soc. Japan* **58**(3) (2006), 617–641. doi:10.2969/jmsj/1156342030.

[22] S.P. Thampi, A. Doostmohammadi, T.N. Shendruk, R. Golestanian and J.M. Yeomans, Active micromachines: Microfluidics powered by mesoscale turbulence, *Science Advances* **2**(7) (2016), 1501854. doi:10.1126/sciadv.1501854.

[23] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North Holland, 1978.

[24] H.H. Wensink, J. Dunkel, S. Heidenreich, K. Drescher, R.E. Goldstein, H. Löwen and J.M. Yeomans, Meso-scale turbulence in living fluids, *Proc. Natl. Acad. Sci. USA* **109** (2012), 14308–14313. doi:10.1073/pnas.1202032109.

[25] K.T. Wu, J.B. Hishamunda, D.T.N. Chen, S.J. DeCamp, Y.W. Chang, A. Fernando-Nieves, S. Fraden and Z. Dogic, Transition from turbulent to coherent flows in confined three-dimensional active fluids, *Science* **355** (2017), 1284.

[26] F. Zanger, H. Löwen and J. Saal, Analysis of a living fluid continuum model, in: *Mathematics for Nonlinear Phenomena – Analysis and Computation*, Y. Maekawa and S. Jimbo, eds, Springer Proceedings in Mathematics and Statistics, Springer International Publishing, 2017, pp. 285–303, Chap. 14. doi:10.1007/978-3-319-66764-5_14.