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# Analytical properties of a (Wannier) exciton-phonon system: On the exclusion of self-trapping and overscreening

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We perform a critical analysis of the concept of phonon-induced phase transitions in connection with (Wannier) exciton-phonon systems. Particular attention is paid to the frequently discussed phenomena of self-trapping and overscreening. We demonstrate for a large class of (generalized Fröhlich) models that a phase-transition-like behavior can be excluded: The ground-state energy, ground-state wave function, and the formal free energy are smooth functions of the electron-phonon and electron-hole coupling parameters. Analogous results hold for a magnetoexciton-phonon system, moving unrestricted or, e.g., within a quantum well. Our proof makes use of functional-analytic and functional-integral methods, which have been successfully applied in previous studies of polaron systems.

## I. INTRODUCTION AND STATEMENT OF PROBLEMS

The purpose of this paper is to clarify the analytical properties of an exciton-phonon system and to resolve some related problems, which have controversially been discussed for a longer time. Our considerations are based on the model Hamiltonian due to Toyozawa[1] and Haken;[2] that is,

$$H := H_P + H_{\mathrm{Ph}} + H_I \ , \tag{1}$$

where

$$H_P := \sum_{n=1}^{2} \mathbf{p}_n^2/2m_n + V(\mathbf{r}_1 - \mathbf{r}_2) \ , \tag{2}$$

$$H_{\mathrm{Ph}} := \int d^3k \ \hbar\omega(\mathbf{k}) a^*(\mathbf{k}) a(\mathbf{k}) \ , \tag{3}$$

$$H_I = \sqrt{\alpha} \sum_{n=1}^{2} (-1)^n \int d^3k [g(\mathbf{k}) a(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{r}_n) + \mathrm{H.c.}] \ . \tag{4}$$

In Eq. (1) the indices $P$, Ph, and $I$ indicate "particles," "phonons," and "interaction," respectively. In Eq. (2) we characterize the particles by momenta $\mathbf{p}_n$, positions $\mathbf{r}_n$, and masses $m_n$. Equation (3) introduces the free-phonon part of the Hamiltonian: $\mathbf{k}$, $\omega(\mathbf{k})$, $a^*(\mathbf{k})$, and $a(\mathbf{k})$ are the wave vector, dispersion, and creation and annihilation operators for the phonons of interest; throughout this paper we assume $\omega(\mathbf{k}) = \omega(-\mathbf{k})$ and $g(\mathbf{k}) = g(-\mathbf{k})$. Finally, the interaction term is given by Eq. (4). The reader will notice that $H_I$ describes a dipole-field interaction, with no mass dependence appearing: the electron and hole constitute an electric dipole, and the phonons a scalar field. For later use we have extracted the familiar electron-phonon–coupling parameter $\sqrt{\alpha}$, $\alpha \geq 0$, from the coupling function $g(\mathbf{k})$.

It is obvious that $H$ is a generalized Fröhlich Hamiltonian; the two-particle potential $V(\mathbf{r}_1 - \mathbf{r}_2)$, the phonon dispersion $\omega(\mathbf{k})$, and the coupling $g(\mathbf{k})$ are hitherto unspecified. Of course, a first choice would be

$$V(\mathbf{r}_1 - \mathbf{r}_2) = -\lambda/|\mathbf{r}_1 - \mathbf{r}_2| \ ; \tag{5}$$

that is, a Coulomb potential. The standard Wannier-exciton-phonon problem is additionally defined by the conditions

$$\omega(\mathbf{k}) = \omega = \mathrm{const}$$

and (6)

$$\sqrt{\alpha}\, g(\mathbf{k}) = \sqrt{\hbar\omega d}\, /2\pi k \ ,$$

where $d$ is fixed as

$$d = (1 - \epsilon_\infty/\epsilon_0)\lambda \tag{7}$$

(see, e.g., Pollmann and Büttner,[3] or Bednarek, Adamowski, and Suffczynski,[4] providing involved variational calculations and further references). Our treatment, however, is not restricted to these special cases. One reason is that we intend to generalize our results (see Sec. IV). It should be possible, for example, to include central-cell corrections in $V(\mathbf{r})$—after all, it is an approximation to replace the total electron-hole potential by the Coulomb expression (5). A second reason to have $V(\mathbf{r})$ and $g(\mathbf{k})$ unspecified (as long as possible) is that the formal structure of the more general theory proves to be more transparent.

To display all relevant physical parameters, we shall frequently write

$$V(\mathbf{r}) = \lambda U(\mathbf{r}) \ , \tag{8}$$

introducing a second coupling constant $\lambda$ besides $\alpha$. If not explicitly stated otherwise, we assume $\lambda \geq 0$ and $U(\mathbf{r}) \leq 0$, and therefore $V(\mathbf{r}) \leq 0$. Abbreviating the two mass parameters available as $\mathbf{m} := (m_1, m_2)$, a complete parametrization of the Hamiltonian is

$$H \equiv H(\alpha, \lambda, \mathbf{m}) . \tag{9}$$

In the following sections we shall entirely be concerned with the analytical behavior of physical quantities (ground-state energy and wave function, formal free energy, etc.) as functions of the parameters $\alpha, \lambda, \mathbf{m}$. A particularly interesting question is whether or not these functions can have points of nonanalyticity. In fact, a positive (negative) answer to this question will decide whether the frequently used concept of phonon-induced phase transitions is appropriate (inappropriate) in connection with exciton-phonon systems.

We have to complete our notational preparations before we can discuss this point in some more detail. Let us introduce center-of-mass and relative coordinates $\mathbf{R}, \mathbf{P}$ and $\mathbf{r}, \mathbf{p}$ as usual, $M$ and $\mu$ being the total and reduced mass. Then we find

$$H_P = \mathbf{P}^2/2M + \mathbf{p}^2/2\mu + \lambda U(\mathbf{r}) , \tag{10}$$

$$H_I = \sqrt{\alpha} \sum_{n=1}^{2} (-1)^n \int d^3k \, [g(\mathbf{k}) a(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{R} + i\gamma_n \mathbf{k}\cdot\mathbf{r}) + \text{H.c.}] , \tag{11}$$

where $\gamma_1 := m_2/M$ and $\gamma_2 := -m_1/M$. Of course, $H_{\text{Ph}}$ remains unchanged. One may verify that $H$ commutes with the operator

$$\mathbf{P}_{\text{tot}} := \mathbf{P} + \int d^3k \, \hbar\mathbf{k} a^*(\mathbf{k}) a(\mathbf{k}) =: \mathbf{P} + \mathbf{P}_{\text{Ph}} \tag{12}$$

of total momentum. This property can profitably be used to eliminate the center-of-mass coordinate from $H$, as was demonstrated by Lee, Low, and Pines:[5] Defining the unitary transformation

$$U := \exp\left[ -\frac{i}{\hbar} \mathbf{R} \cdot \mathbf{P}_{\text{Ph}} \right] \tag{13}$$

and calculating $H' := U^{-1} H U$ as well as $\mathbf{P}'_{\text{tot}} := U^{-1} \mathbf{P}_{\text{tot}} U$, we arrive at

$$\mathbf{P}'_{\text{tot}} = \mathbf{P} , \tag{14}$$

$$H' = H_P + H_{\text{Ph}} + H'_I , \tag{15}$$

where

$$H'_P = (\mathbf{P} - \mathbf{P}_{\text{Ph}})^2/2M + \mathbf{p}^2/2\mu + \lambda U(\mathbf{r}) , \tag{16}$$

$$H'_I = \sqrt{\alpha} \sum_{n=1}^{2} (-1)^n \int d^3k \, [g(\mathbf{k}) a(\mathbf{k}) \exp(i\gamma_n \mathbf{k}\cdot\mathbf{r}) + \text{H.c.}] . \tag{17}$$

The conservation of total momentum is now equivalent to $[H', \mathbf{P}] = 0$ and permits us to restrict $H'$ to the subspace of eigenfunctions of $\mathbf{P}$ with some eigenvalue $\hbar\mathbf{Q}$. We are thus led to

$$H'(\mathbf{Q}) := (\hbar\mathbf{Q} - \mathbf{P}_{\text{Ph}})^2/2M + \mathbf{p}^2/2\mu + \lambda U(\mathbf{r}) + H_{\text{Ph}} + H'_I . \tag{18}$$

Clearly, $H'(\mathbf{Q})$ no longer depends on the center-of-mass variables. Moreover, it is sufficient to discuss $H'(\mathbf{Q})$ instead of $H$ or $H'$. Let us now turn to the ground-state energy $E(\mathbf{Q}, \alpha, \lambda, \mathbf{m})$ of $H'(\mathbf{Q})$—that is, the lower limit of the spectrum of $H'(\mathbf{Q})$. The reader will notice that the existence of $E(\mathbf{Q}, \alpha, \lambda, \mathbf{m})$ is directly connected with a proper mathematical definition of $H'(\mathbf{Q})$, which, in turn, presupposes the specification of admissible dispersions $\omega(\mathbf{k})$, couplings $g(\mathbf{k})$, and potentials $V(\mathbf{r})$. We discuss this point in full detail in Secs. II and III. For the moment, we take the existence of the ground-state energy for granted. Even more, we assume $E(\mathbf{Q}, \alpha, \lambda, \mathbf{m})$ to be a simple eigenvalue of $H'(\mathbf{Q})$, the corresponding eigenfunction being $\psi(\mathbf{Q}, \alpha, \lambda, \mathbf{m})$. In view of our central topic we state the following.

*Problem 1: What is the domain of analyticity of $E(\boldsymbol{Q}, \alpha, \lambda, \boldsymbol{m})$ and $\psi(\boldsymbol{Q}, \alpha, \lambda, \boldsymbol{m})$ as functions of $\boldsymbol{Q}, \alpha, \lambda, \boldsymbol{m}$?*

Our solution of problem 1 will make extensive use of functional-analytic theorems. An alternative and, in part, complementary approach to spectral properties of Hamiltonians is possible by means of functional-integral techniques. In connection with exciton physics, we refer to Haken,[6] Moskalenko,[7] Schultz,[8] and Adamowski, Gerlach, and Leschke.[9] Let us introduce the diagonal element of the reduced density operator; that is,

$$\rho(\alpha, \beta, \lambda, \mathbf{m}) := \text{tr}_{\text{Ph}} \langle \mathbf{r}_1, \mathbf{r}_2 | e^{-\beta H} | \mathbf{r}_1, \mathbf{r}_2 \rangle . \tag{19}$$

Here, $\text{tr}_{\text{Ph}}$ indicates the trace operation with respect to phonons, $\beta > 0$ is a formal inverse temperature, and $\mathbf{r}_n$ is the position of particle $n$. We remark that the right-hand side of Eq. (19) depends on the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$; nevertheless, we skip this dependence in $\rho$, as it is irrelevant for our considerations.

It proves useful to relate $\rho(\alpha, \beta, \lambda, \mathbf{m})$ to the readily accessible expression $\rho(0, \beta, 0, \mathbf{m})$ for a (hypothetical) free-electron-hole pair and to define a formal partition function

$$Z(\alpha, \beta, \lambda, \mathbf{m}) := \rho(\alpha, \beta, \lambda, \mathbf{m}) / \rho(0, \beta, 0, \mathbf{m}) . \tag{20}$$

On one hand, $Z$ can conveniently be expressed by a functional integral, namely

$$Z(\alpha, \beta, \lambda, \mathbf{m}) = \langle \exp(-S_I - S_\lambda) \rangle , \tag{21}$$

where the expectation value of a quantity $A$ is defined as

$$\langle A \rangle := \frac{\int \delta^3 R_1 \int \delta^3 R_2 \, A[\mathbf{R}_1, \mathbf{R}_2] \exp(-S_0[\mathbf{R}_1, \mathbf{R}_2])}{\int \delta^3 R_1 \int \delta^3 R_2 \exp(-S_0[\mathbf{R}_1, \mathbf{R}_2])} . \tag{22}$$

In (22) the integration is of Wiener-Feynman type and has to be performed over all closed, real paths $\mathbf{R}_n(\tau)$ with $\mathbf{R}_n(0) = \mathbf{R}_n(\beta) = \mathbf{r}_n$. In particular, one has

$$S_0[\mathbf{R}_1,\mathbf{R}_2]:=\int_0^\beta d\tau \sum_{n=1}^{2} m_n \dot{\mathbf{R}}_n^2(\tau)/2 \ , \tag{23}$$

$$S_I[\mathbf{R}_1,\mathbf{R}_2]:=-\alpha \sum_{n,n'=1}^{2} (-1)^{n+n'} \int d^3k |g(\mathbf{k})|^2 \int_0^\beta d\tau \int_0^\beta d\tau' \, G_{\omega(\mathbf{k})}(\tau-\tau') \exp\{i\mathbf{k}\cdot[\mathbf{R}_n(\tau)-\mathbf{R}_{n'}(\tau')]\}$$

$$=: \sum_{n,n'=1}^{2} s_{nn'}[\mathbf{R}_1,\mathbf{R}_2] \ , \tag{24}$$

$$S_\lambda[\mathbf{R}_1,\mathbf{R}_2]:=\lambda \int_0^\beta d\tau \, U(\mathbf{R}_1(\tau)-\mathbf{R}_2(\tau)) \ . \tag{25}$$

$G_\omega(\tau)$ is the temperature-dependent Green function of a harmonic oscillator and is given by

$$G_\omega(\tau):=\cosh(\beta\hbar\omega/2-\hbar\omega|\tau|)/[2\sinh(\beta\hbar\omega/2)] \ . \tag{26}$$

On the other hand, $Z(\alpha,\beta,\lambda,\mathbf{m})$ is connected with the formal free energy $F(\alpha,\beta,\lambda,\mathbf{m})$ as follows:

$$Z(\alpha,\beta,\lambda,\mathbf{m}) =: \exp\{-\beta[F(\alpha,\beta,\lambda,\mathbf{m})-F(0,\beta,0,\mathbf{m})]\} \ . \tag{27}$$

From $F$ we can derive all spectral properties by familiar manipulations. We state the following.

*Problem 2: Is $F(\alpha,\beta,\lambda,m)-F(0,\beta,0,m)$ a real analytic function of $\alpha,\beta,\lambda,m$ for $0\le\alpha<\infty$, $0<\beta<\infty$, $0\le\lambda<\infty$, and $0<m_n<\infty$?*

Having posed the problems, we close this section with some comments on the phase-transition concept in connection with exciton-phonon interactions. Starting from expressions (10) and (11) for the Hamiltonian $H$, one realizes that the total problem incorporates aspects of free- and bound-polaron motion: The center-of-mass–coordinate part mimics the former property and the relative-coordinate part the latter. Accordingly, several possibilities for an eventual nonanalytical behavior of $E(\mathbf{Q},\alpha,\lambda,\mathbf{m})$, $\psi(\mathbf{Q},\alpha,\lambda,\mathbf{m})$, and $F(\alpha,\beta,\lambda,\mathbf{m})$ have been discussed.

In analogy to the free-polaron case, center-of-mass motion was supposed to show a delocalization-localization transition, mostly denoted a self-trapping phenomenon (as for this idea in general, we refer to Landau's early paper[10] on polarons and the extensions of Rashba[11] and Toyozawa[12] for excitons). The numerical work in favor of this hypothesis is entirely of variational type and related to the standard model, defined in Eqs. (5)–(7). The decisive conclusions are as follows (see, e.g., Pekar, Rashba, and Sheka[13] and Shimamura and Matsuura[14]): There exists a finite value $\lambda=\lambda_c$ such that the ground-state wave function of $H$ is delocalized for $\lambda<\lambda_c$ and localized for $\lambda>\lambda_c$ (we insert as a remark that only one coupling parameter appears in the standard case; $\alpha$ depends on $\lambda$ and the electron mass). The corresponding upper bound for the energy exhibits a nonanalyticity for $\lambda=\lambda_c$. Surprisingly enough, these variational results proved to be artifacts of the approximations made. The present authors demonstrated in Ref. 15 that the ground-state wave function is always delocalized. We generalize this discussion in Sec. II.

The relative motion can qualitatively be discussed by means of the functional-integral expressions for $S_I$ and $S_\lambda$, given in Eqs. (24) and (25). Both are negative definite: $S_\lambda\le 0$ is a consequence of $\lambda U(\mathbf{r})\le 0$; $S_I\le 0$ can be derived from Eq. (24), if one inserts a Fourier representation of $G_\omega(\tau)$ (see Adamowski, Gerlach, and Leschke in Ref. 16). One should notice, however, that the sign of the electron-hole part of $S_I$, namely $s_{12}+s_{21}=2s_{12}$, is not generally fixed. Insofar as this is concerned, the original electron-hole interaction $S_\lambda$ can partially be weakened or strengthened by the phonon-induced one, provided $g(\mathbf{k})$ is correspondingly chosen. We comment on both alternatives: $s_{12}[\mathbf{R}_1,\mathbf{R}_2]$ can be positive for all paths $\mathbf{R}_n(\tau)$; an important example is $g(\mathbf{k})\propto k^{-1}$. In this case, $2s_{12}$ may even overcompensate $S_\lambda$—the phenomenon of "overscreening" is definitely possible, if $\sqrt{\alpha}g(\mathbf{k})$ and $\lambda U(\mathbf{r})$ are unrelated. If we suppose that $\lambda U(\mathbf{r})$ as such is binding, the appearance of a repulsive, phonon-induced interaction between electron and hole may cause a localization-delocalization transition with respect to relative motion. Consequently, ground-state quantities such as the energy will be nonanalytic functions of the coupling parameters (also see Sumi[17]).

Irrespective of the form of $g(\mathbf{k})$, $s_{12}$ cannot be negative for all paths $\mathbf{R}(\tau)$, but, at most, for a certain subset. This is an important difference in comparison to deformation-potential couplings, which cause an expression $c_n/c_{n'}$ in Eq. (24) instead of $(-1)^{n+n'}$, $c_n$ being a potential coefficient. Then, $s_{12}$ can, in fact, be negative for all paths $\mathbf{R}_n(\tau)$ and will strengthen the potential $\lambda U(\mathbf{r})$ in any case. Sumi[17] claims that under such circumstances a self-shrinking transition can occur. At first glance, the situation reminds one of the bound-polaron system. This does exhibit a nonanalyticity (the so-called pinning transition)—but only for potentials of short-range type (see Spohn[18] and Löwen[19]). Of course, the electron-hole potential is a long-range potential; with this in mind, the assertion of an excitonic self-shrinking transition appears to be doubtful.

Summarizing our introductory discussion, the relevance of problems 1 and 2 should be clear. If self-trapping, self-shrinking, or overscreening transitions occur, they induce a nonanalytical behavior of $E(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ and/or $F(\alpha,\beta,\lambda,\mathbf{m})$. In the following sections we specify conditions which guarantee an analytical behavior of both quantities and exclude any phase-transition-like behavior of the systems under consideration.

The organization of the rest of this paper is as follows. We discuss the ground-state properties in Sec. II and those for finite temperature in Sec. III—for technical reasons, we have to separate both cases. In Sec. IV we provide some extensions of the results in Secs. II and III, and include some comments on related, unsolved problems. We close with a short summary in Sec. V.

## II. ANALYTICITY OF GROUND-STATE ENERGY AND WAVE FUNCTION

The solution of problem 1 is technically involved; before we can list three statements and prove their validity, some remarks are necessary.

The first is concerned with the free-polaron system. In this case the analytical properties of such ground-state quantities as $E(\mathbf{Q},\alpha)$ and $\psi(\mathbf{Q},\alpha)$ have been clarified in a pioneering paper of Fröhlich.[20] If the inequalities

$$\omega(\mathbf{k})\geq\omega>0,\quad \omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)\geq\omega(\mathbf{k}_1+\mathbf{k}_2)$$

and (28)

$$\int d^3k|g(k)|^2/[1+(ak)^2]<\infty$$

are valid ($a:=\sqrt{\hbar/m\omega}$ being the polaron radius), $E(\mathbf{Q},\alpha)$ and $\psi(\mathbf{Q},\alpha)$ are real analytic functions of $\mathbf{Q}$ and $\alpha$ for $\hbar^2\mathbf{Q}^2/2m<\hbar\omega$ and $0\leq\alpha<\infty$. Condition (28) is basic to our discussion also. We anticipate that $\omega(\mathbf{k})\geq\omega>0$ is particularly important, since thereby the existence of an energy gap above the ground-state energy is guaranteed. The latter property, in turn, will prove indispensable to ensuring the analyticity of the exciton quantities $E(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ and $\psi(\mathbf{Q},\alpha,\lambda,\mathbf{m})$.

Our second remark refers to the electron-hole potential $\lambda U(\mathbf{r})$. According to our introductory discussion, we should know about the binding properties of $H_{\rm rel}:=\mathbf{p}^2/2\mu+\lambda U(\mathbf{r})$. This is the case for the so-called Rollnik class $R$ of potentials, defined by the inequality

$$\lambda^2\int d^3r\int d^3r'\,|U(\mathbf{r})|\,|U(\mathbf{r}')|/|\mathbf{r}-\mathbf{r}'|^2<\infty\ . \tag{29}$$

The reader will notice that the left-hand side of (29) is proportional to the Birman-Schwinger bound for the number of bound states of $H_{\rm rel}$. Therefore, the physical significance of $R$ is clear (also see Reed and Simon[21]). Unfortunately, the Rollnik class is too small for our intentions; in fact, it contains the familiar short-range potentials, but not the long-range ones of Coulomb type. To incorporate these too, we consider the extension

$$R':=R+L^\infty_\epsilon(\mathbb{R}^3) \tag{30}$$

of $R$. Any element $\lambda U\in R'$ can be represented as $\lambda U=\lambda U'+\lambda U''$, where $\lambda U'\in R$ and $|\lambda U''|<\epsilon$ for arbitrary $\epsilon>0$. As for a detailed discussion of $R'$, we quote Simon[22] and again Reed and Simon.[21]

Throughout this section we assume $\lambda U\in R'$. To facilitate further reading, we note two relevant properties of $\lambda U$: First, $H_{\rm rel}$ is well defined—in particular, it is self-adjoint and bounded from below in the sense of forms; secondly, $\lambda U$ is infinitesimally form-bounded with respect to $\mathbf{p}^2/2\mu$; that is,

$$|\langle\psi|\lambda U|\psi\rangle|\leq\epsilon\langle\psi|\mathbf{p}^2/2\mu|\psi\rangle+b(\epsilon)\langle\psi|\psi\rangle \tag{31}$$

for any $\epsilon>0$ and some positive $b(\epsilon)$.

Finally, we add a remark on the continuum edge $E_c(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ of the Hamiltonian $H'(\mathbf{Q})$ under consideration [to be explicit, we shall write $H'(\mathbf{Q})\equiv H'(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ if necessary]. Intuitively, we can imagine two types of continuum states: On one hand, the electron and hole could be sufficiently separated such as to "escape" from the potential $\lambda U(\mathbf{r})$. Consequently, the energy of relative motion would become continuous. Defining

$$E_c^1(\mathbf{Q},\alpha,\mathbf{m}):=\inf\sigma_{\rm ess}[H'(\mathbf{Q},\alpha,0,\mathbf{m})]\ , \tag{32}$$

where $\sigma_{\rm ess}[H']$ denotes the essential part of the spectrum of $H'$, we expect $E_c^1$ to be one upper bound on $E_c$. On the other hand, a ground-state exciton of energy $E(\mathbf{Q}-\mathbf{k},\alpha,\lambda,\mathbf{m})$ might absorb a phonon of energy $\hbar\omega(\mathbf{k})$, thereby reaching a continuum state too. If we define

$$E_c^2(\mathbf{Q},\alpha,\lambda,\mathbf{m}):=\inf_{\mathbf{k}}[E(\mathbf{Q}-\mathbf{k},\alpha,\lambda,\mathbf{m})+\hbar\omega(\mathbf{k})]\ , \tag{33}$$

we should obtain a second upper bound on $E_c$. We shall subsequently prove that

$$E_c(\mathbf{Q},\alpha,\lambda,\mathbf{m})\geq\min\{E_c^1(\mathbf{Q},\alpha,\mathbf{m}),\,E_c^2(\mathbf{Q},\alpha,\lambda,\mathbf{m})\} \tag{34}$$

is true, providing us with a lower bound on $E_c$. It is very probable that (34) is an equality rather than an inequality. In any case, this inequality is sufficient to proceed.

We are now prepared to make our first statement. Considering the alternative $E_c^2\leq E_c^1$ in inequality (34), we state the following.

*Statement 1a: Assume the validity of (28) for the dispersion $\omega(\mathbf{k})$ and the coupling $g(\mathbf{k})$; let $\lambda U(\mathbf{r})$ be negative and an element of $R'$. Furthermore, $H_{rel}$ should have at least one bound state for any $\lambda>0$, the corresponding energy being strictly negative. Finally, let $E_c^2\leq E_c^1$. Then, $E(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ exists and is an isolated, simple eigenvalue of $H'(\mathbf{Q})$ for $0\leq\alpha<\infty$, $0<\lambda<\infty$, $0<m_n<\infty$, and $\mathbf{Q}$ in a certain surrounding of $\mathbf{Q}=0$. $E(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ as well as $\psi(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ are real analytic functions of $\mathbf{Q},\alpha,\lambda,\mathbf{m}$ in the specified domain.*

We believe that this statement is even true for $\hbar^2Q^2/2M<\hbar\omega$, which would be in accordance with the free-polaron result, mentioned before. Our proof, however, will be restricted to a smaller-$Q$ regime.

Let us now turn to the case $E_c^2>E_c^1$. Here, the existence of a discrete ground state depends sensitively on the specific form of $\omega(\mathbf{k})$ as well as $g(\mathbf{k})$ and $\lambda U(\mathbf{r})$. Recalling our introductory discussion, the overscreening phenomenon has to be excluded; of course, this is in general impossible, but has to be done as an additional presupposition. We find the following.

*Statement 1b: As for $\omega(\mathbf{k})$, $g(\mathbf{k})$, and $\lambda U(\mathbf{r})$, assume the conditions of statement 1a. Consider the case $E_c^2>E_c^1$ and suppose additionally that the ground-state energy is an ei-*

*genvalue of $H'(Q)$ for $Q$ below some positive constant. Then, $E(Q,\alpha,\lambda,m)$ is a simple eigenvalue of $H'(Q)$ for $0 \le \alpha < \infty$, $0 < \lambda < \infty$, $0 < m_n < \infty$, and $Q$ in a certain surrounding of $Q=0$. $E(Q,\alpha,\lambda,m)$, as well as $\psi(Q,\alpha,\lambda,m)$, are real analytic functions of $Q,\alpha,\lambda,m$ in the specified domain.*

At first glance, this statement appears to be a rather formal one. To correct such an impression, we stress two points: It is not at all trivial that the existence of a ground-state eigenvalue $E(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ is sufficient to guarantee its nondegeneracy and analyticity as function of $\alpha,\lambda,\mathbf{m}$. Furthermore, statement 1b is well suited for direct applications: In many cases the existence of a ground-state eigenvalue can be established by variational methods. The standard Wannier-exciton-phonon problem as defined in Eqs. (5)–(7) is an illustrative example. Here, we arrive at the more specific statement that follows.

*Statement 1c: Consider the standard exciton-phonon problem. Then, the ground-state energy $E(Q,\lambda,m)$ exists and is an isolated, simple eigenvalue for $0<\lambda<\infty$, $0<m_n<\infty$, and $Q$ in a certain surrounding of $Q=0$. $E(Q,\lambda,m)$ and $\psi(Q,\lambda,m)$ are real analytic functions of $\lambda$, $m$, and $Q$ in the specified domain.*

We list some direct consequences of these statements: Many ground-state observables can be calculated as derivatives of $E(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ with respect to one of the parameters, or as expectation values of the type $\langle \psi(\mathbf{Q},\alpha,\lambda,\mathbf{m})|X|\psi(\mathbf{Q},\alpha,\lambda,\mathbf{m})\rangle$, where $X$ is an operator independent of $\mathbf{Q},\alpha,\lambda,\mathbf{m}$. As examples, we mention the total mass of the polaronic exciton, its radius, or the mean phonon number associated with the state $\psi(\mathbf{Q},\alpha,\lambda,\mathbf{m})$. Statements 1a–1c demonstrate that all these quantities are smooth functions of $\mathbf{Q},\alpha,\lambda,\mathbf{m}$ in the surrounding of $\mathbf{Q}=0$. The smoothness of the total mass is of particular interest. If a self-trapping transition would occur for a certain critical value $\alpha=\alpha_c$, the total mass should become infinite. This is excluded—in complete agreement with the previous result that the ground-state wave function of $H$ is always delocalized (see Ref. 15).

We mentioned before that some authors found a self-trapping transition for the standard system or one under the conditions of statement 1a (see, e.g., Refs. 13 and 14). The weak point of their arguments is that they have to rely on approximate (variational) expressions for the energy. Without underestimating the merits of the variational results as such, the nonanalyticities have to be classified as artifacts of the approximations made.

We are now going to prove the above statements. It may be helpful to indicate the basic steps of our discussion. First, we shall fix the position of the continuum edge $E_c(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ of $H'(\mathbf{Q},\alpha,\lambda,\mathbf{m})$. Secondly, we shall show that the ground-state energy $E(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ exists and is below $E_c(\mathbf{Q},\alpha,\lambda,\mathbf{m})$. Thirdly, we have to guarantee the nondegeneracy of the ground state. If this can be taken for granted, we finally apply analytical perturbation theory to assure the analyticity of, e.g., $E(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ as a function of $\mathbf{Q},\alpha,\lambda,\mathbf{m}$. We shall make extensive use of the quoted work of Fröhlich[20] and the extensions of Löwen.[19,23] Particularly important for this work is the treatment of a polaron in an external potential, which can be found in Ref. 19; if details of that discussion can be totally transferred, we shall not repeat them here.

Let us begin with the first step as enumerated above and fix the continuum edge of $H'(\mathbf{Q})$. First, we introduce a large-$k$ (or uv) cutoff in the coupling $g(\mathbf{k})$ as described in any of the pertinent references.[19,20,23] This assures us that the resulting cutoff Hamiltonian $\tilde{H}'(\mathbf{Q})$ is self-adjoint and bounded from below. Secondly, we redefine $\tilde{H}'(\mathbf{Q})$ on a discrete phonon-momentum lattice. In a first step we admit only a finite number of phonon modes, the corresponding Hamiltonian being $\hat{H}'(\mathbf{Q})$. Clearly, the essential spectrum of $\hat{H}'(\mathbf{Q})$ can only originate from the relative motion of the electron and hole. In fact, the application of Weyl's essential spectrum theorem proves that Eq. (32) yields the correct continuum edge, if $\hat{H}'(\mathbf{Q})$ is inserted instead of $H'(\mathbf{Q})$ (see Refs. 19 and 21). We mention that here the condition $\lambda U \in R'$ is important. Finally, we have to remove the momentum lattice and the large-$k$ cutoff. The corresponding procedures are well established and can be traced back to a paper of Nelson.[24] Admitting continuous $\mathbf{k}$ values, one creates new, phonon-induced continuum states. Using the property $\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)\ge\omega(\mathbf{k}_1+\mathbf{k}_2)$, one can show that the corresponding continuum edge is produced by one-phonon states and bounded from below by $E_c^2$ according to Eq. (33), but still evaluated for $\tilde{H}'(\mathbf{Q})$ instead of $H'(\mathbf{Q})$. The remaining large-$k$ cutoff can be removed by a canonical transformation, which is well known as the "oscillator transform"; the technical details can be found in Ref. 19. We note that in this context the inequalities (28) are essential.

In summary, we have established use of relation (34) for finding the position of the continuum edge $E_c$. To prove the existence of eigenstates of $H'(\mathbf{Q})$ with energies below $E_c$, we use two inequalities, namely

$$E(\mathbf{0},\alpha,\lambda,\mathbf{m}) \le E(\mathbf{Q}\ne\mathbf{0},\,\alpha,\lambda,\mathbf{m})\ , \tag{35}$$

$$E(\mathbf{Q},\alpha,\lambda,\mathbf{m}) \le E(\mathbf{0},\alpha,\lambda,\mathbf{m})+\hbar^2\mathbf{Q}^2/2M\ . \tag{36}$$

Equation (35) can be shown in analogy to the free-polaron case; we recall that the $\mathbf{Q}$ dependence of the Hamiltonians of interest is exactly the same (see, e.g., Gerlach, Löwen, and Schliffke in Ref. 25). To demonstrate the validity of (36), it proves useful to revert to the cutoff Hamiltonian $\tilde{H}'(\mathbf{Q})$ and to discuss

$$\tilde{H}'(\mathbf{Q})-\hbar^2\mathbf{Q}^2/2M =: h-\hbar\mathbf{Q}\cdot\mathbf{P}_{\mathrm{Ph}}/M\ , \tag{37}$$

where $h$ is independent of $\mathbf{Q}$. As $\mathbf{P}_{\mathrm{Ph}}$ is infinitesimally form-bounded with respect to $h$, the ground-state energy of $h-\hbar\mathbf{Q}\cdot\mathbf{P}_{\mathrm{Ph}}/M$ is monotonically decreasing and continuous in $\mathbf{Q}$ (see Ref. 21, p. 98). Eliminating the large-$k$ cutoff as before, we arrive at (36).

To use relations (35) and (36), let us discuss the alternative $E_c^2 \le E_c^1$ in (34). In this case inequality (34) can indeed be replaced by the equality $E_c=E_c^2$; the proof of theorem IV in Ref. 19 can be directly transferred to the present problem. In any case, expression (33) for $E_c^2$, $\omega(\mathbf{k}) \geq \omega > 0$, and inequality (35) assure us that

$$E_c(\mathbf{Q},\alpha,\lambda,\mathbf{m}) \geq E(\mathbf{0},\alpha,\lambda,\mathbf{m}) + \hbar\omega \tag{38}$$

is true. Contrasting this result with (36), we find

$$E(\mathbf{Q},\alpha,\lambda,\mathbf{m}) < E_c(\mathbf{Q},\alpha,\lambda,\mathbf{m}) \quad \text{for } \hbar^2\mathbf{Q}^2/2M < \hbar\omega \ . \tag{39}$$

Therefore, the existence of a (discrete) eigenvalue of $H'(\mathbf{Q})$ is established in the quoted $\mathbf{Q}$ domain.

According to our outline for the proof, the next property to show is the nondegeneracy of $E(\mathbf{Q},\alpha,\lambda,\mathbf{m})$. To do so, it is sufficient to demonstrate that $\exp[-tH'(\mathbf{Q})]$ is positivity improving for $t > 0$ (see Reed and Simon,[21] p. 204). A corresponding proof exists for the Hamiltonian $H'$ according to Eq. (15); $H'$ differs from $H'(\mathbf{Q})$ only insofar as the operator $\mathbf{P}$ appears instead of the eigenvalue $\hbar\mathbf{Q}$. Returning to the discussion of $H'$ in Ref. 15, this can be directly transferred to $H'(\mathbf{Q})$, if $\mathbf{Q}=\mathbf{0}$. Consequently, $E(\mathbf{0},\alpha,\lambda,\mathbf{m})$ is a simple eigenvalue. Because of the continuity with respect to $\mathbf{Q}$, the same holds true for $E(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ in a certain surrounding of $\mathbf{Q}=\mathbf{0}$ and $0 \leq \alpha < \infty$, $0 < \lambda < \infty$, and $0 < m_n < \infty$.

We now turn to analytical perturbation theory as developed by Kato (see, e.g., Ref. 26). As an example, we discuss the $\lambda$ dependence in $E(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ and $\psi(\mathbf{Q},\alpha,\lambda,\mathbf{m})$: Comparing different values of $\lambda$, one derives

$$H'(\mathbf{Q},\alpha,\lambda,\mathbf{m}) - H'(\mathbf{Q},\alpha,\lambda_0,\mathbf{m}) = (\lambda-\lambda_0)U \ . \tag{40}$$

As $\lambda U$ is an element of $R'$, $U$ is form-bounded with respect to the "unperturbed" term $H'(\mathbf{Q},\alpha,\lambda_0,\mathbf{m})$ (we recall that the relative bound is even zero). Consequently, the Hamiltonians $H'(\mathbf{Q},\alpha,\lambda,\mathbf{m})$—$\lambda$ considered a variable and $\mathbf{Q},\alpha,\mathbf{m}$ being fixed—form a holomorphic family of self-adjoint operators (type $B$) in the sense of Kato. If we combine this property with the nondegeneracy of $E(\mathbf{Q},\alpha,\lambda,\mathbf{m})$, we can conclude that $E(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ and $\psi(\mathbf{Q},\alpha,\lambda,\mathbf{m})$ are real analytic functions of $\lambda$ in the quoted domain.

Using analogous arguments for the $\mathbf{Q}$, $\alpha$, and $\mathbf{m}$ dependences, we can complete the proof of statement 1a.

In view of the above discussion, our reasoning with respect to statement 1b can be shortened: We presupposed the existence of a discrete eigenstate of $H'(\mathbf{Q})$ in a surrounding of $\mathbf{Q}=\mathbf{0}$; starting from that point, we can repeat the arguments of the previous proof.

We finally turn to statement 1c: As the case $E_c^2 \leq E_c^1$ is completely covered by statement 1a, we are left with $E_c^2 > E_c^1$. Recalling the functional-integral expression (24) for $S_I$ and inequality (35), we derive for a model with coupling $g(\mathbf{k}) \propto 1/k$ (as in the standard case)

$$E(\mathbf{Q},\alpha,0,\mathbf{m}) \geq \sum_{n=1}^{2} E(\alpha,m_n) \ , \tag{41}$$

where $E(\alpha,m_n)$ is the (hypothetical) free-polaron ground-state energy of constituent $n$ of the exciton. Therefore,

$$E_c^1(\mathbf{0},\alpha,\mathbf{m}) \geq \sum_{n=1}^{2} E(\alpha,m_n) \tag{42}$$

is true. On the other hand, the ground-state energy $E(\mathbf{0},\lambda,\mathbf{m})$ of the standard exciton-phonon problem was shown to fulfill

$$E(\mathbf{0},\lambda,\mathbf{m}) < \sum_{n=1}^{2} E(\alpha,m_n) \tag{43}$$

for $0 < \lambda < \infty$ and $0 < m_n < \infty$ (see Adamowski, Gerlach, and Leschke[9]). Therefore, $E(\mathbf{0},\lambda,\mathbf{m})$ is a (discrete) eigenvalue.

After this is established, we proceed again as in the case of statement 1a and complete the proof.

## III. ANALYTICITY OF THE FORMAL FREE ENERGY

The central result of this part can be summarized as follows.

*Statement 2: Take the existence of $F(\alpha,\beta,\lambda,\mathbf{m})-F(0,\beta,0,\mathbf{m})$ or, equivalently, the boundedness of $Z(\alpha,\beta,\lambda,\mathbf{m})$ for granted: $0<Z(\alpha,\beta,\lambda,\mathbf{m})<\infty$. Then, $F(\alpha,\beta,\lambda,\mathbf{m})-F(0,\beta,0,\mathbf{m})$ is a real analytic function of $\alpha,\beta,\lambda,\mathbf{m}$. Sufficient conditions for the existence of $F(\alpha,\beta,\lambda,\mathbf{m})-F(0,\beta,0,\mathbf{m})$ in $0 \leq \alpha < \infty$, $0 < \beta < \infty$, $0 \leq \lambda < \infty$, and $0 < m_n < \infty$ are*

$$|\underline{U}(\mathbf{k})| < c/k^2 \tag{44}$$

*[$\underline{U}(\mathbf{k})$ being the Fourier transform of $U(\mathbf{r})$ and c some positive constant], and*

$$\int d^3k \, |g(\mathbf{k})|^2/\omega(\mathbf{k}) < \infty \quad \textit{(short-range case)} \ , \tag{45}$$

*or*

$$\omega(k) \geq \omega > 0, \quad |g(\mathbf{k})| < \text{const} \times k^{-1} \quad \textit{(long-range case)} \ . \tag{46}$$

The preceding statement is surprisingly general, one reason being the simple $\alpha$, $\beta$, $\lambda$, and $\mathbf{m}$ dependences of the actions $S_0$, $S_I$, and $S_\lambda$. Furthermore, the hard part of the proof is to establish that $Z$ is bounded. Fortunately, the latter problem can be reduced to the corresponding one for a free polaron, which has been solved previously (see Ref. 27).

Our proof starts with a discussion of Eq. (21) for $Z$. We truncate the exponential function and introduce

$$Z_{NM}(\alpha,\beta,\lambda,\mathbf{m}) := \left\langle \sum_{n=0}^{N} \frac{1}{n!}(-S_I)^n \sum_{m=0}^{M} \frac{1}{m!}(-S_\lambda)^m \right\rangle = \sum_{n=0}^{N} \frac{1}{n!} \sum_{m=0}^{M} \frac{1}{m!} \langle (-S_I)^n (-S_\lambda)^m \rangle \ . \tag{47}$$

Direct inspection of $S_I$ and $S_\lambda$ shows that we can write

$$\langle(-S_I)^n(-S_\lambda)^m\rangle =: \alpha^n\lambda^m f_{nm}(\beta,\mathbf{m}) , \tag{48}$$

where $f_{nm}(\beta,\mathbf{m})$ is positive because of the positivity of $-S_I$ and $-S_\lambda$. Furthermore, the functional integral in $\langle(-S_I)^n(-S_\lambda)^m\rangle$ can be evaluated according to the recipes given in Ref. 28 [one has to insert a Fourier integral for $U(\mathbf{r})$, leading to functional integrals of Gaussian type]. $f_{nm}(\beta,\mathbf{m})$ proves to be a finite-dimensional integral with a real analytic dependence of $\beta$ and $\mathbf{m}$—provided the integral exists at all. For the moment, we take this for granted. We obtain the following properties of the finite series $Z_{NM}(\alpha,\beta,\lambda,\mathbf{m})$: First, this quantity is strictly positive and monotonically increasing as function of $N,M$ ($\alpha,\beta,\lambda,\mathbf{m}$ fixed); secondly, it is a real analytic function of $\alpha,\beta,\lambda,\mathbf{m}$ for $0\le\alpha<\infty$, $0\le\lambda<\infty$, $0<\beta<\infty$, and $0<m_n<\infty$ ($N,M$ fixed).

Let us begin with the first property: If we can prove that $Z_{NM}(\alpha,\beta,\lambda,\mathbf{m})$ is uniformly bounded from above by some function $C(\alpha,\beta,\lambda,\mathbf{m})<\infty$, the monotone convergence theorem assures us that

$$\lim_{N\to\infty}\lim_{M\to\infty} Z_{NM}(\alpha,\beta,\lambda,\mathbf{m})=Z_{\infty\infty}(\alpha,\beta,\lambda,\mathbf{m})$$

exists, and

$$Z(\alpha,\beta,\lambda,\mathbf{m})=Z_{\infty\infty}(\alpha,\beta,\lambda,\mathbf{m}) = \sum_{n,m=0}^{\infty}\frac{\alpha^n}{n!}\frac{\lambda^m}{m!}f_{nm}(\beta,\mathbf{m}) \tag{49}$$

holds true [clearly, this includes the existence of any single term $f_{nm}(\beta,\mathbf{m})$, which was in question until now]. On the other hand, we may assume the existence of $Z(\alpha,\beta,\lambda,\mathbf{m})$. In that case, we certainly have $Z>Z_{NM}$ and (49) is valid again. $Z(\alpha,\beta,\lambda,\mathbf{m})$ exists if and only if this is true for $Z_{\infty\infty}(\alpha,\beta,\lambda,\mathbf{m})$.

Therefore, we examine (49) in more detail. Despite its introduction as a function of positive $\alpha,\beta,\lambda,m_n$, the right-hand side of (49) may be discussed as an infinite series of complex $\alpha,\beta,\lambda,m_n$. This series converges absolutely for all $\alpha$, $\lambda$, $0<\mathrm{Re}\beta<\infty$, and $0<\mathrm{Re}m_n<\infty$, and uniformly on every compact parameter set, if (and only if) the original series (49) converges for $0\le\alpha<\infty$, $0\le\lambda<\infty$, $0<\beta<\infty$, and $0<m_n<\infty$. Combining this fact and the analyticity of $f_{nm}(\beta,\mathbf{m})$ for $0<\mathrm{Re}\beta<\infty$ and $0<\mathrm{Re}m_n<\infty$, we may state the following result: The right-hand side of Eq. (49) exists as a complex series for all $\alpha$, $\lambda$, $0<\mathrm{Re}\beta<\infty$, and $0<\mathrm{Re}m_n<\infty$, and represents an analytical function in this domain, if it exists as a real series for $0\le\alpha<\infty$, $0\le\lambda<\infty$, $0<\beta<\infty$, and $0<m_n<\infty$. As the latter is strictly positive, we have proven the first part of statement 2.

The remaining task is to establish the convergence of the real series (49) under the conditions (44) and (45) or (44) and (46) as given in statement 2. To do so, we derive from Eq. (47)

$$Z_{NM}(\alpha,\beta,\lambda,\mathbf{m})\le\frac{1}{2}\left\langle\left[\sum_{n=0}^{N}\frac{1}{n!}(-S_I)^n\right]^2+\left[\sum_{m=0}^{M}\frac{1}{m!}(-S_\lambda)^m\right]^2\right\rangle$$
$$\le\frac{1}{2}\left[\left\langle\sum_{n=0}^{2N}\frac{1}{n!}(-2S_I)^n\right\rangle+\left\langle\sum_{m=0}^{2M}\frac{1}{m!}(-2S_\lambda)^m\right\rangle\right] . \tag{50}$$

According to (24), $-S_I[\mathbf{R}_1,\mathbf{R}_2]$ is bounded from above as follows:

$$-S_I[\mathbf{R}_1,\mathbf{R}_2]\le-2\sum_{n=1}^{2}S_{nn}[\mathbf{R}_1,\mathbf{R}_2] . \tag{51}$$

Inequality (51) provides us with the upper bound

$$\left\langle\sum_{n=0}^{2N}\frac{1}{n!}(-2S_I)^n\right\rangle\le\left\langle\sum_{n=0}^{2N}\frac{1}{n!}(-4s_{11})^n\right\rangle\left\langle\sum_{n=0}^{2N}\frac{1}{n!}(-4s_{22})^n\right\rangle \tag{52}$$

for the first term on the right-hand side of (50). We have studied this bound in Ref. 27 and have found that it does converge under condition (45) or (46).

Finally, we are left with the second term in (50), containing the potential. Inserting the explicit representation of $S_\lambda$ [Eq. (25)] and $U(\mathbf{r})$ as Fourier integral, we have to evaluate functional integrals of Gaussian type. As all of these are positive, we can use condition (44) within the integral to find

$$\left\langle\sum_{m=0}^{2M}\frac{1}{m!}(-2S_\lambda)^m\right\rangle\le\left\langle\exp\left[\frac{2\lambda c}{(2\pi)^3}\int_0^\beta d\tau\int d^3k k^{-2}\exp\{i\mathbf{k}\cdot[\mathbf{R}_1(\tau)-\mathbf{R}_2(\tau)]\}\right]\right\rangle \tag{53}$$

uniformly in $M$. As the right-hand side does exist (see Ref. 27), statement 2 is proven.

## IV. EXTENSIONS

The previously used methods are well suited to treat certain generalizations of the Hamiltonian $H$ according to Eq. (1). In other cases, we are still left with partially unsolved problems. We give some examples for both alternatives.

For a comparison with experiments, it may be necessary to include an excitonic coupling to several phonon branches. One ensures—via direct inspection of our proofs—that statements 1a–2 can correspondingly be generalized, if the quoted conditions are fulfilled by every branch.

A particularly involved and unsolved question is whether $\omega(\mathbf{k})\geq\omega>0$ is necessary to guarantee an analytical behavior of ground-state quantities. We are not aware of any analyticity proof admitting an acoustical dispersion. Obviously, this is in marked contrast to the finite-temperature case, where a zero of $\omega(\mathbf{k})$ is unimportant, provided inequality (45) is valid. We remark in addition that localization studies for the free acoustical polaron do exist and have proved the absence of a delocalization-localization transition (see Spohn in Ref. 18). It should be possible to transfer these results to the center-of-mass motion of an exciton.

Interesting new problems can be found if one tries to discuss a nonparabolic band structure $\epsilon(\mathbf{p})$ instead of $\mathbf{p}^2/2m$. The total momentum is still conserved and permits us to introduce a momentum-decomposed Hamiltonian $H'(\mathbf{Q})$ in complete analogy to Eq. (18). Moreover, the discussion of the continuum edge can be transferred. Provided we can establish the existence of (discrete) eigenstates of $H'(\mathbf{Q})$, the nondegeneracy of the ground state can be guaranteed for $\mathbf{Q}=\mathbf{0}$, if $\exp[-t\epsilon(\mathbf{p})]$ has a positive Fourier transform (see Ref. 15). It may now be a harder task than above to ensure the applicability of analytical perturbation theory; if an application is possible, statements 1a–1c will hold again.

As another extension, we provide a brief discussion of a magnetoexciton-phonon system. There exists an enormous number of publications on the corresponding one-particle system, usually denoted a magnetopolaron (see, e.g., Peeters and Devreese[29] and Löwen[23]).

What about the analytical properties of a polaronic exciton in a homogeneous magnetic field? The generalization of Hamiltonian (1) is straightforward. Introducing center-of-mass and relative coordinates as before, and using a symmetrical gauge for the vector potential, one now realizes that

$$\hat{\mathbf{P}}_{\text{tot}}:=\mathbf{P}+\mathbf{P}_{\text{Ph}}-|e|\mathbf{B}\times\mathbf{r}/2 \tag{54}$$

is a conserved quantity. One can unitarily transform $\hat{\mathbf{P}}_{\text{tot}}$ and the original Hamiltonian into

$$\mathbf{P}_{\text{tot}}=\mathbf{P}+\mathbf{P}_{\text{Ph}}\ , \tag{55}$$

$$\begin{aligned}H:=&(-\gamma_2\mathbf{P}+\mathbf{p}+|e|\mathbf{B}\times\mathbf{r}/2)^2/2m_1\\&+(\gamma_1\mathbf{P}-\mathbf{p}+|e|\mathbf{B}\times\mathbf{r}/2)^2/2m_2\\&+\lambda U(\mathbf{r})+H_{\text{Ph}}+H_I\ ,\end{aligned} \tag{56}$$

where $H_{\text{Ph}}$ and $H_I$ are defined in Eqs. (3) and (11). The decisive point is that again a Lee-Low-Pines procedure can be used to obtain a momentum-decomposed Hamiltonian $H'(\mathbf{Q})$. Combining the methods of this article and of Ref. 23, one can generalize statements 1a–1c for nonzero magnetic field.

The discussion of the finite-temperature properties is even simpler. One has to recall that a nonzero magnetic field causes an additional, purely imaginary and bilinear action $S_B$ within the total action, $B$ appearing as a simple prefactor (see, e.g., Ref. 29). Instead of Eq. (21), we find

$$Z(\alpha,\beta,\lambda,\mathbf{m},B):=\langle\exp(-S_I-S_\lambda-S_B)\rangle\ . \tag{57}$$

As $S_B$ is purely imaginary, we can immediately derive that

$$|Z(\alpha,\beta,\lambda,\mathbf{m},B)|\leq Z(\alpha,\beta,\lambda,\mathbf{m},0) \tag{58}$$

is true, providing us with an upper bound on $Z(\alpha,\beta,\lambda,\mathbf{m},B)$. The further discussion can totally be transferred from above. We conclude the following: Under the conditions quoted for the statements from above, the magnetoexciton-phonon system exhibits no nonanalyticities. We can slightly generalize the last example to get a qualitative model for a magnetoexciton-phonon system in a quantum-well structure. Assuming the wells to be parallel to the $x$-$y$ plane, we describe the well influence by an additional potential $\lambda'U'(z)$ for electron and hole. Let us suppose that $\lambda'U'(z)\leq 0$ and $U'\in L^2(\mathbb{R})$ is true. Then, we can correspondingly generalize the above analyticity conclusions.

We close this part with a brief remark concerning discrete models. In our context, they are necessary to describe Frenkel excitons. It is beyond the scope of this article to analyze the corresponding literature. We stress, however, that similar discussions of analytical properties have been performed, leading to analogous results (as for a reference, we quote Löwen[30]).

## V. SUMMARY

The intention of this article was to analyze the possibilities for a phase-transition-like behavior in exciton-phonon systems. For the class of generalized Fröhlich models, no such possibility exists: Neither self-trapping nor overscreening can occur, provided the conditions for statements 1a–2 are fulfilled. Our results complement perfectly similar ones for free and bound polarons, as well as polarons in external homogeneous fields.

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